

We count the number of possible functions f with input set $[k] = \{1, 2, \dots, k\}$ and output set $[n] = \{1, 2, \dots, n\}$, subject to restrictions (injective or surjective). We may picture f as a way of distributing k balls (marked $1, \dots, k$) into n baskets (marked $1, \dots, n$). A map is injective if each basket contains at most one ball, or surjective if no basket is empty.

Indistinguishable $[k]$ means we consider two functions the same whenever they differ by a permutation of the inputs $[k]$; so we picture the k balls as identical, unmarked. Similarly, *indistinguishable* $[n]$ means we consider classes of functions up to permutation of the outputs $[n]$, so we picture the n baskets as identical and movable, and we cannot distinguish a first basket, second basket, etc.

$f : [k] \rightarrow [n]$		ALL FUNCTIONS	INJECTIONS ($k \leq n$)	SURJECTIONS ($k \geq n$)
DIST	DIST	$\textcircled{1}$ n^k $n^k = n \cdot n^{k-1}$	$\textcircled{2}$ n^k $n^k = (n-k+1) n^{k-1}$	$\textcircled{3}$ $\text{surj}(k, n) = \sum_{i=0}^n (-1)^i \binom{n}{i} (n-i)^k$ $\text{surj}(k, n) = n \text{surj}(k-1, n-1) + n \text{surj}(k-1, n)$
IND	DIST	$\textcircled{4}$ $\binom{\binom{n}{k}}{k} = \frac{n^k}{k!}$ $\binom{\binom{n}{k}}{k} = \binom{\binom{n}{k-1}}{k-1} + \binom{\binom{n-1}{k}}{k}$	$\textcircled{5}$ $\binom{n}{k} = \frac{n^k}{k!}$ $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$	$\textcircled{6}$ $a(k, n) = \binom{\binom{n}{k-n}}{k-n} = \binom{k-1}{n-1}$ $a(k, n) = a(k-1, n-1) + a(k-1, n)$
DIST	IND	$\textcircled{7}$ $\left\{ \begin{matrix} k \\ \leq n \end{matrix} \right\} = \left\{ \begin{matrix} k \\ 1 \end{matrix} \right\} + \left\{ \begin{matrix} k \\ 2 \end{matrix} \right\} + \dots + \left\{ \begin{matrix} k \\ n \end{matrix} \right\}$ $\left\{ \begin{matrix} k \\ \leq n \end{matrix} \right\} = \sum_{i=0}^{k-1} \binom{k-1}{i} \left\{ \begin{matrix} i \\ \leq n-1 \end{matrix} \right\}$	$\textcircled{8}$ 1	$\textcircled{9}$ $\left\{ \begin{matrix} k \\ n \end{matrix} \right\} = \frac{\text{surj}(k, n)}{n!}$ $\left\{ \begin{matrix} k \\ n \end{matrix} \right\} = \left\{ \begin{matrix} k-1 \\ n-1 \end{matrix} \right\} + n \left\{ \begin{matrix} k-1 \\ n \end{matrix} \right\}$
IND	IND	$\textcircled{10}$ $p_{\leq n}(k) = p_1(k) + p_2(k) + \dots + p_n(k)$ $p_{\leq n}(k) = p_{\leq n-1}(k) + p_{\leq n}(k-n)$	$\textcircled{11}$ 1	$\textcircled{12}$ $p_n(k) = p_{\leq n}(k-n)$ $p_n(k) = p_{n-1}(k-1) + p_n(k-n)$

- Binomial coefficient, n choose k , $\binom{n}{k}$: $(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$. Multiset, n multi-choose k , $\binom{n}{k}$: $\frac{1}{(1-x)^n} = \sum_{k \geq 0} \binom{n}{k} x^k$, $\binom{n}{k} = \binom{n+k-1}{k}$.
- Stirling partition number (2nd kind) $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$: $y^n = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} y^k$, $y^n = \sum_{k=1}^n (-1)^{n-k} \binom{n}{k} y^k$, $\sum_{n \geq 0} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{x^n}{n!} = \frac{1}{k!} (e^x - 1)^k$.
- Stirling cycle num (1st kind) $\left[\begin{matrix} n \\ k \end{matrix} \right] = \# \text{ perms } w \in S_n \text{ with } k \text{ cycles}$, $\left[\begin{matrix} n \\ k \end{matrix} \right] = \left[\begin{matrix} n-1 \\ k-1 \end{matrix} \right] + (n-1) \left[\begin{matrix} n-1 \\ k \end{matrix} \right]$, $\sum_{n \geq 0} \left[\begin{matrix} n \\ k \end{matrix} \right] \frac{x^n}{n!} = \log^k(\frac{1}{1-x})$, $\sum_{n,k \geq 0} \left[\begin{matrix} n \\ k \end{matrix} \right] y^k \frac{x^n}{n!} = \frac{1}{(1-x)^y}$.
- Bell number $B_k = \left\{ \begin{matrix} k \\ \leq k \end{matrix} \right\} = \left\{ \begin{matrix} k \\ 1 \end{matrix} \right\} + \left\{ \begin{matrix} k \\ 2 \end{matrix} \right\} + \dots + \left\{ \begin{matrix} k \\ k \end{matrix} \right\}$. Recurrence: $B_k = \sum_{i=0}^{k-1} \binom{k-1}{i} B_i$. Dobinski: $B_k = \frac{1}{e} \sum_{i \geq 0} \frac{i^k}{i!}$. $\sum_{k \geq 0} B_k \frac{x^k}{k!} = \exp(e^x - 1)$.
- Partition number $p(k) = p_{\leq k}(k) = p_1(k) + p_2(k) + \dots + p_k(k)$: $\sum_{k \geq 0} p(k) x^k = \prod_{i \geq 1} \frac{1}{1-x^i}$, Hardy-Ramanujan: $p(k) \sim \frac{1}{4n\sqrt{3}} \exp(\pi \sqrt{\frac{2n}{3}})$. Euler: $\prod_{i \geq 0} (1-x^i) = 1 + \sum_{m=1}^{\infty} (-1)^m (x^{m(3m-1)/2} + x^{m(3m+1)/2}) \iff p(k) = \sum_{m=1}^{\infty} (-1)^{m+1} (p(k-\frac{1}{2}m(3m-1)) + p(k-\frac{1}{2}m(3m+1)))$.
- Fibonacci number $F_k = F_{k-1} + F_{k-2}$ from $F_0 = 0$, $F_1 = 1$. Binet: $F_k = \frac{1}{\sqrt{5}} (\phi^k - (-\psi)^k) = \text{round}(\frac{\phi^k}{\sqrt{5}})$, where $\phi = \frac{\sqrt{5}+1}{2}$, $\psi = \frac{\sqrt{5}-1}{2}$.
- Catalan number $C_k = \sum_{i=0}^{k-1} C_i C_{k-i}$ from $C_0 = 1$; $C_k = \frac{1}{k+1} \binom{2k}{k} = \# \text{ binary ordered trees (2k+1 nodes)}$; $\# \text{ ordered trees (k nodes)}$.
- Derangement number (perms with no fixed points) $D_k = k! (1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^k}{k!})$. $\sum_{k \geq 0} D_k \frac{x^k}{k!} = \frac{e^{-x}}{1-x}$. $D_k = (k-1)(D_{k-1} + D_{k-2})$.
- Cayley: labeled, unordered trees $T_k = k^{k-2}$. Unlabeled, unordered rooted trees r_k : $\sum_{k \geq 1} r_k x^k = x \prod_{i \geq 1} \frac{1}{(1-x^i)^{ri}}$; $r_{k+1} = \frac{1}{k} \sum_{j=1}^k \sum_{i|j} i r_i r_{k-j+1}$.
- Euler number $E_n = \# \pi \in S_n$ alternating perms $\pi(1) > \pi(2) < \pi(3) > \dots$. $E_{n+1} = \frac{1}{2} \sum_{i=0}^n \binom{n}{i} E_i E_{n-i}$. $\sum_{k \geq 0} E_k \frac{x^k}{k!} = \tan(x) + \sec(x)$.