Math 880 **The Twelvefold Way** **Fall 2016**

We consider combinatorial classes $\mathcal{F}$ of functions f with domain $[k] = \{1, 2, \ldots, k\}$ and codomain $[n] = \{1, 2, \ldots, n\}$, subject to restrictions on f (injective or surjective). We may picture f as a way of distributing k balls into n baskets.

We say $[k]$ is indistinguishable if we consider mappings up to symmetry by S_k; and similarly for $[n]$ indistinguishable. By convention there is a unique surjection $f : \{\} \to \{\}$, and a unique injection $f : \{\} \to [n]$ for all n.

Each function $f : [k] \to [n]$ has size $|f| = k$ and weight $||f|| = n$, and contributes the term $\frac{x^k}{k!} t^n$; and we have bigraded counting sequence $F_{k,n}$. We give a combinatorial construction for $\tilde{\mathcal{F}}$ (if $[k]$ is distinguishable) or $\mathcal{F}$ (if $[k]$ is indistinguishable), identifying a function f with its sequence of pre-image sets $(S_1, \ldots, S_n)$, where $S_i = f^{-1}(i)$. The elements of the sets are built from the basic class $[1] = \{1\}$ with $|1| = 1$ and $||1|| = 0$; while each set is marked by t^1 using a factor $\{\emptyset\}$ with $|\emptyset| = 0$ and $||\emptyset|| = 1$. We give the corresponding bivariate exponential generating function $\tilde{F}(x,t) = \sum_{k,n\geq 0} \tilde{F}_{k,n} \frac{x^k}{k!} t^n$, or the bivariate ordinary generating function $F(x,t) = \sum_{k,n\geq 0} F_{k,n}\, x^n u^n$. We also give the univariate function $F_n(x) = [t^n] F(x,t)$.

$f : [k] \to [n]$	ALL FUNCTIONS	INJECTIONS	SURJECTIONS
DIST DIST	n^k $\mathrm{Seq}(\emptyset_t \times \mathrm{Set}\ \widetilde{[1]_x})$ $\tilde{F}(x,t) = \dfrac{1}{1 - ue^x}$ $\tilde{F}_n(x) = e^{kx}$	$n^{\underline{k}}$ $\mathrm{Seq}(\emptyset_t \times \mathrm{Set}_{\leq 1}\ \widetilde{[1]_x})$ $\tilde{F}(x,t) = \dfrac{1}{1 - t(1+x)}$ $\tilde{F}_n(x) = (1+x)^n$	$n!\left\{{n \atop k}\right\}$ $\mathrm{Seq}(\emptyset_t \times \mathrm{Set}_{\geq 1}\ \widetilde{[1]_x})$ $\tilde{F}(x,t) = \dfrac{1}{1 - t(e^x - 1)}$ $\tilde{F}_n(x) = (e^x - 1)^n$
IND DIST	$\left(\!\!\binom{k}{n}\!\!\right) = \mathrm{comp}_n^{\geq 0}(k)$ $\mathrm{Seq}(\emptyset_t \times \mathrm{Seq}\ [1]_x)$ $F(x,t) = \dfrac{1}{1 - \frac{t}{1-x}}$ $F_n(x) = \dfrac{1}{(1-x)^n}$	$\binom{k}{n} = \mathrm{comp}_n^{\leq 1}(k)$ $\mathrm{Seq}(\emptyset_t \times \mathrm{Seq}_{\leq 1}\ [1]_x)$ $F(x,t) = \dfrac{1}{1 - t(1+x)}$ $F_n(x) = (1+x)^n$	$\left(\!\!\binom{n}{k-n}\!\!\right) = \binom{k-1}{n-1} = \mathrm{comp}_n^{\geq 1}(k)$ $\mathrm{Seq}(\emptyset_t \times \mathrm{Seq}_{\geq 1}\ [1]_x)$ $F(x,t) = \dfrac{1}{1 - \frac{tx}{1-x}}$ $F_n(x) = \dfrac{x^n}{(1-x)^n}$
DIST IND	$\left\{{n \atop 1}\right\} + \cdots + \left\{{n \atop k}\right\}$ $\mathrm{Set}(\emptyset_t \times \mathrm{Set}\ \widetilde{[1]_x})$ $\tilde{F}(x,t) = \sum_{n \geq \ell \geq 0} \frac{1}{\ell!}(e^x - 1)^\ell t^n$ $\tilde{F}_n(x) = \sum_{\ell=0}^{n} \frac{1}{\ell!}(e^x - 1)^\ell$	$\begin{cases} 1 & \text{if } k \leq n \\ 0 & \text{else} \end{cases}$ $\mathrm{Set}(\emptyset_t \times \mathrm{Set}_{\leq 1}\ \widetilde{[1]_x})$ $\tilde{F}(x,t) = \sum_{n \geq k \geq 0} \frac{1}{k!} x^k t^n$ $\tilde{F}_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k$	$\left\{{n \atop k}\right\}$ $\mathrm{Set}(\emptyset_t \times \mathrm{Set}_{\geq 1}\ \widetilde{[1]_x})$ $\tilde{F}(x,t) = e^{t(e^x - 1)}$ $\tilde{F}_n(x) = \frac{1}{n!}(e^x - 1)^n$
IND IND	$p_{\leq n}(k)$ $\mathrm{MSet}(\emptyset_t \times \mathrm{Seq}\ [1]_x)$ $F(x,t) = \prod_{j \geq 0} \dfrac{1}{1 - tx^j}$ $F_n(x) = \prod_{j=1}^{n} \dfrac{1}{1 - x^j}$	$\begin{cases} 1 & \text{if } k \leq n \\ 0 & \text{else} \end{cases}$ $\mathrm{MSet}(\emptyset_t \times \mathrm{Seq}_{\leq 1}\ [1]_x)$ $F(x,t) = \dfrac{1}{(1-t)(1-tx)}$ $F_n(x) = \dfrac{1 - x^{n+1}}{1 - x}$	$p_n(k) = p_{\leq n}(k-n)$ $\mathrm{MSet}(\emptyset_t \times \mathrm{Seq}_{\geq 1}\ [1]_x)$ $F(x,t) = \prod_{j \geq 1} \dfrac{1}{1 - tx^j}$ $F_n(x) = x^n \prod_{j=1}^{n} \dfrac{1}{1 - x^j}$

Some remarks on the entries of the table.

- In the first two columns, the functions $F(x,t)$ do not specialize to formal power series at $t=1$, because there are infinitely many objects of size $k=0$, namely the injections $f:\{\}\to[n]$ for $n\geq 0$.
- Row IND DIST: Here $\mathrm{comp}_n^S(k)$ counts the n-compositions of k by elements of a set $S\subset\mathbb{Z}_{\geq 0}$, i.e. lists $(k_1,\ldots,k_n)$ with $k_1+\cdots+k_n=k$ and $k_i\in S$.
- Row DIST IND: For the first two columns, the exponential formula does not hold for the outer Set operation, because we are applying Set to a class containing elements of size zero (i.e. $\emptyset_t$). The objects counted are of the form $\{S_1,\ldots,S_\ell,\{\},\cdots,\{\}\}$ with $n-\ell$ empty sets, and ℓ non-empty sets $S_1\coprod\cdots\coprod S_\ell=[k]$. Now the symmetric group S_n does *not* act freely on these n-tuples because of the $n-\ell$ identical empty sets. Rather, we must fix ℓ and mod out by S_ℓ, resulting in the formulas given.
- Row IND IND: The usual multiset formulas do not take account of the weight n, so we must analyze the cases directly using the product principle. Choosing a partition means determining the number of 0's, the number of 1's, etc.; and choosing ℓ with multiplicity m corresponds to the monomial $(tx^\ell)^m$.

  To determine the partial generating functions $F_n(x)$, we use the transpose bijection: a partition $k=\lambda_1+\cdots+\lambda_n$ with exactly n parts $\lambda_i\geq 0$ corresponds to a partition $k=\mu_1+\mu_2+\cdots$ with any number of parts μ_j restricted by $1\leq\mu_j\leq n$.