Let T_n be the number of rooted, unlabelled trees with n vertices: $T_0 = 0$, $T_1 = T_2 = 1$, $T_3 = 2$, $T_4 = 4$. For $n = 4$, the tree can be either: a linear path rooted at an end or an internal vertex; or a star rooted at the center or at a leaf; giving $T_4 = 4$ distinct choices.

The combinatorial specification $\mathcal{T} = \{\bullet\} \times \text{MSet}(\mathcal{T})$ gives the following equation involving the generating function $T(x) = \sum_{n\geq 1} T_n x^n$:

$$T(x) = x \prod_{j\geq 1} \frac{1}{(1-x^j)^{T_j}}\,.$$

We will apply logarithmic differentiation to obtain an amazing recurrence for T_n. Writing out the equation as:

$$\sum_{n\geq 0} T_{n+1}x^n = \prod_{j\geq 1}(1-x^j)^{-T_j}\,,$$

we apply the operation $x\frac{d}{dx}\log$ to both sides. On the left side, the identity $x\frac{d}{dx}\log f(x) = xf'(x)/f(x)$ implies:

$$x\frac{d}{dx}\log \sum_{n\geq 0} T_{n+1}x^n = \frac{\sum_{n\geq 1} nT_{n+1}x^n}{\sum_{m\geq 0} T_{m+1}x^m}\,.$$

On the right side, we use $\log(ab) = \log a + \log b$ and $\log(a^b) = b\log a$ to get:

$$x\frac{d}{dx}\log \prod_{j\geq 1}(1-x^j)^{-T_j} = \sum_{j\geq 1} -T_j\, x\frac{d}{dx}\log(1-x^j)$$

$$= \sum_{j\geq 1} T_j \frac{jx^j}{1-x^j} = \sum_{j\geq 1}\sum_{i\geq 1} jT_j\, x^{ij} = \sum_{k\geq 1}\Big(\sum_{j|k} jT_j\Big)x^k\,.$$

where in the last equality we substitute $k = ij$, and $j|k$ means j divides k.

Now equating the two sides, clearing the denominator, and collecting x^n terms, we get:

$$\sum_{n\geq 1} nT_{n+1}x^n = \sum_{k\geq 1}\Big(\sum_{j|k} jT_j\Big)x^k \cdot \sum_{m\geq 0} T_{m+1}x^m = \sum_{n\geq 1}\Big(\sum_{k=1}^{n}\sum_{j|k} jT_jT_{n-k+1}\Big)x^n$$

where in the second equality we substitute $n = k+m$, so that $m+1 = n-k+1$. We conclude:

$$T_{n+1} = \frac{1}{n}\sum_{k=1}^{n}\sum_{j|k} jT_jT_{n-k+1}\,,$$

where the right side involves only $T_1, \ldots, T_n$. This recurrence has no combinatorial explanation, but it is fairly efficient computationally.

Example: To compute T_5, we sum over $k = 1, 2, 3, 4$ and j running over all divisors of k: that is, $(j, k) = (1, 1)$, (1,2), (2,2), (1,3), (3,3), (1,4), (2,4), (4,4), so that:

$$\begin{aligned} T_5 &= \tfrac{1}{4}(T_1T_4 + T_1T_3 + 2T_2T_3 + T_1T_2 + 3T_3T_2 + T_1T_1 + 2T_2T_1 + 4T_4T_1) \\ &= \tfrac{1}{4}(4+2+4+1+6+1+2+16) = 9\,. \end{aligned}$$