**Math 880** **Homework 4** **Fall 2023**

Feel free to discuss homework problems with other students, and to learn from references, but please do not look up specific answers. Write out solutions in your own words, and give explicit credit for any significant help.

NOTES. For a poset $(\mathcal{P}, \leq)$ with minimal element $\hat{0}$, we define the *incidence algebra*:

$$I(\mathcal{P}) \ = \ \{\alpha : \mathrm{Int}(\mathcal{P}) \to \mathbb{C}\} \ \cong \ \textstyle\bigoplus_{a\leq b} \mathbb{C}[a,b],$$

functions on $\mathrm{Int}(\mathcal{P})$, the set of intervals $[a,b]$ with $a \leq b$, under convolution product:

$$(\alpha\cdot\beta)[a,b] = \sum_{a\leq c\leq b} \alpha[a,c]\,\beta[c,b], \qquad [a,b]\cdot[c,d] = \begin{cases} [a,d] & \text{if } b=c \\ 0 & \text{otherwise.} \end{cases}$$

We can realize $I(\mathcal{P})$ inside upper-triangular matrices in $M_{n\times n}(\mathbb{C})$, where $n = \#\mathcal{P}$.

Our posets are often semi-infinite, and contain standard elements $\hat{0}, \hat{1}, \hat{2}, \ldots$, and there is a natural equivalence relation $[a,b] \sim [c,d]$ which splits $\mathrm{Int}(\mathcal{P})$ into equivalence classes $\bar{0}, \bar{1}, \bar{2}, \ldots$, where $\bar{n}$ is the equivalence class of $[\hat{0}, \hat{n}]$. We define the *reduced incidence algebra*:

$$R(\mathcal{P}) = \{\, \alpha \in I(\mathcal{P}) \text{ with } \alpha[a,b] = \alpha[c,d] \text{ for } [a,b]\sim[c,d] \,\} \ = \ \bigoplus_{n=0}^{\infty} \mathbb{C}\,\bar{n}.$$

We may identify an element $\alpha \in R(\mathcal{P})$ with a function $\bar{\alpha} : \mathcal{P} \to \mathbb{C}$ with $\bar{\alpha}(a) = \alpha(\bar{n})$ where $[\hat{0}, a] \sim \bar{n}$. Indeed, we can denote $\alpha$ as a kind of generating function for the sequence $a_n = \alpha(\bar{n}) = \bar{\alpha}(\hat{n})$:

$$\alpha = a_0\bar{0} + a_1\bar{1} + a_2\bar{2} + \cdots .$$

$R(\mathcal{P})$ contains the identity or delta-function $\delta[a,a] = 1$, $\delta[a,b] = 0$ for $a < b$; the zeta function $\zeta[a,b] = 1$, or $\zeta = \sum_{n=0}^{\infty} \bar{n}$; and the Möbius function $\mu = \zeta^{-1}$.

For a *binomial poset*, a ranked poset with $\hat{0} \lessdot \hat{1} \lessdot \hat{2} \lessdot \cdots$, such that every interval $[a,b]$ with length $\mathrm{rk}(b) - \mathrm{rk}(a) = n$ has $B(n)$ maximal chains, the algebra $R(\mathcal{P})$ is isomorphic to the formal power series ring $\mathbb{C}[[x]]$, with $\bar{n} \cong x^n/B(n)$, so we can write functions as $\alpha \cong a_0 + a_1 x + a_2 \frac{x^2}{B(2)} + a_3 \frac{x^3}{B(3)} + \cdots$ for $a_n = \alpha(\bar{n})$.

For a finite field $F = \mathbb{F}_q$, consider the poset $\mathcal{B}_n(q)$ of linear subspaces $V \subset F^n$ ordered by inclusion, a $q$-analog of the Boolean poset $\mathcal{B}_n \cong \wp[n]$ of subsets $I \subset [n]$. The union of these spaces via the inclusions $0 \subset F^1 \subset F^2 \subset \cdots$ is the binomial poset $\mathcal{P} = \mathcal{B}(q)$, with standard elements $\hat{n} = F^n$. The factorial function is:

$$B(n) = \#\mathrm{Flag}(F^n) = [n]_q^! = [n]_q[n{-}1]_q \cdots [1]_q\,, \quad [n]_q = \#\mathbb{P}(F^n) = \tfrac{q^n-1}{q-1} = 1{+}q{+}\cdots{+}q^{n-1}.$$

The reduced incidence algebra $R(\mathcal{P})$ corresponds to *Eulerian generating functions*:

$$f(x) \ = \ \sum_{n\geq 0} a_n \frac{x^n}{[n]_q^!} \ = \ a_0 + a_1 x + a_2 \frac{x^2}{1{+}q} + a_3 \frac{x^3}{(1{+}q{+}q^2)(1{+}q)} + \cdots .$$

Problems

**1.** The formula $\zeta\cdot\mu = \delta$ is equivalent to the equations: $\mu[a,a] = 1$ and $\sum_{a\leq c\leq b}\mu[a,c] = 0$ for $a < b$. The direct product of two posets is $\mathcal{P}\times\mathcal{Q}$, with $(p,q)\leq(p',q')$ whenever $p\leq p'$ and $q\leq q'$. Prove that the Möbius function of the product is the product of the individual Möbius functions of $\mathcal{P}$, $\mathcal{Q}$:

$$\mu_{\mathcal{P}\times\mathcal{Q}}[(p,q),(p',q')] \;=\; \mu_{\mathcal{P}}[p,p']\;\mu_{\mathcal{Q}}[q,q'].$$

**2.** For the poset $\mathcal{P} = \mathcal{D}_{18}$, the 6-element poset of divisors of $18 = 2\cdot 3^2$ ordered by divisibility, work out the Möbius function $\mu[a,b]$ in several ways:

**a.** Write a $6{\times}6$ matrix $Z$ corresponding to $\zeta[a,b] = 1$ for all $a\leq b$, and invert by Gaussian elimination: write a double matrix $[Z\,|\,I\,]$, then row reduce to the form $[\,I\,|\,M]$, so that $M = Z^{-1}$.

**b.** Write $Z = I + N$, for identity $I$ and strictly upper-triangular $N$, nilpotent with $N^6 = 0$. By computer, expand geometric series $M = (I+N)^{-1} = I - N + N^2 - \cdots$.

**c.** For each $a\in\mathcal{P}$, draw a copy of the Hasse diagram (a $2\times 3$ rectangle). Mark $\mu[a,a] = 1$, then work upwards computing $\mu[a,b]$ using the recurrence $\mu[a,b] = -\sum_{a\leq c<b}\mu[a,c]$.

**d.** Apply the product formula of #1 above to $\mathcal{D}_{18}\cong[2]\times[3]$, the direct product of two chains. Match this with Mobius' original description: $\mu[d,n] = (-1)^k$ if $n/d$ is the product of $k$ distinct primes, and $\mu[d,n] = 0$ if $n/d$ is divisible by a square number.

**e.** Evaluate Phillip Hall's Formula: $\mu[a,b] = \hat{c}_0 - \hat{c}_1 + \hat{c}_2 - \cdots$, where $\hat{c}_d$ is the number of chains of length $d$ from $a$ to $b$ in $\mathcal{P}$, starting with $\hat{c}_0 = 0, \hat{c}_1 = 1$.

**f.** Consider $\mathcal{P} = \mathcal{Q}\sqcup\{\hat{0},\hat{1}\}$, where $\mathcal{Q} = \{a\in\mathcal{P}$ with $\hat{0} < a < \hat{1}\}$, and form the simplicial complex $\Delta(Q)$ whose elements are all chains in $\mathcal{Q}$. Draw a picture of the corresponding topological space: one-simplexes glued at their endpoints.

Hall's Formula says $\mu[\hat{0},\hat{1}] = \tilde{\chi}(\Delta(Q))$, the reduced Euler characteristic of the above topological space, the alternating sum of the number of simplexes of each dimension, minus 1. Compute $\tilde{\chi}(\Delta(Q))$ from this definition. Also, find the simplest triangulation of this space, and compute $\tilde{\chi}$ from that.

**3.** The posets $\mathcal{D}_n$ of divisors of $n$ have the semi-infinite union $\mathcal{P} = \mathcal{D}_\infty = \{1, 2, 3, \ldots\}$ ordered by divisibility. This has standard elements $\hat{n} = n$, and the equivalence of intervals $[a, b] \sim [c, d]$ whenever $b/a = d/c$, which induces equivalence classes $\bar{n} = \overline{[1, n]}$, making a basis of the reduced algebra $R(\mathcal{P}) = \bigoplus_{n \geq 1} \mathbb{C}\, \bar{n}$. We have $\bar{n}\bar{m} = \overline{nm}$, so $R(\mathcal{P})$ embeds in the ring of complex functions via $\bar{n} \cong n^{-s}$, and $\alpha \in R(\mathcal{P})$ corresponds to a Dirichlet series $\sum_{n \geq 1} \frac{\alpha(\bar{n})}{n^s}$, where $s$ is a complex variable.

Now recall how we count necklaces of $n$ beads chosen from $k$ colors, orbits of the cyclic symmetry group $G = C_n$. Since $G$ has $\phi(n/d)$ permutations with $d$ cycles, Burnside's Theorem gives the number of orbits as the necklace polynomial:

$$N_n(k) = \tfrac{1}{\#G} \sum_{\pi \in G} k^{\mathrm{cyc}(\pi)} = \tfrac{1}{n} \sum_{d \mid n} \phi(n/d)\, k^d.$$

**a.** The *Moreau polynomial* $M_n(k)$ couts the *aperiodic* necklaces, those with no cyclic symmetry, so that their orbit has size $n$. First show the convolution formula:

$$k^n = \sum_{d \mid n} d\, M_d(k).$$

That is, if we consider $\alpha(n) = k^n$ and $\beta(n) = nM_n(k)$ as elements of $R(\mathcal{P})$, we have $\alpha = \beta \cdot \zeta$. Now give a summation formula for $M_n(k)$ via Mobius inversion.

**b.** Similarly, give a formula for $M_n(k)$ as a summation in terms of $N_d(k)$.

**4.** The zeta function of the $q$-Boolean poset $\mathcal{P} = \mathcal{B}(q)$ is $\zeta = \exp_q(x) = \sum_{n \geq 0} \frac{x^n}{[n]_q^!}$.

*Theorem:* The reciprocal of $\zeta = \exp_q(x)$ is the power series

$$\mu \;=\; \sum_{n \geq 0} (-1)^n q^{\binom{n}{2}} \frac{x^n}{[n]_q^!}.$$

**a.** Consider the bigraded combinatorial class $\mathcal{E}$ containing pairs $(\lambda, n)$, where $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ is an integer partition of length $n$ and size $|\lambda| = \sum \lambda_j$.

Find bijections to write the generating function $E(q, x) = \sum_{(\lambda, n) \in \mathcal{E}} q^{|\lambda|} x^n$ as:

$$E(q, x) \;=\; \prod_{i \geq 0} \frac{1}{1 - q^i x} \;=\; \sum_{n \geq 0} \frac{x^n}{(q)_n},$$

where $(q)_n = (1-q)(1-q^2) \cdots (1-q^n)$ is the *$q$-Pochhammer symbol.*

*Hint:* For the product expression, write partitions in terms of multiplicities $m_i = \#\{j \mid \lambda_j = i\}$. For the sum expression, consider the Young diagram of $\lambda$ with rows of length $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$ and columns of length $n \geq \lambda'_1 \geq \lambda'_2 \geq \cdots > 0$.

**b.** Prove the following identity in close analogy to part (a):

$$F(q, x) \;=\; \prod_{i \geq 0} (1 - q^i x) \;=\; \sum_{n \geq 0} (-1)^n q^{\binom{n}{2}} \frac{x^n}{(q)_n}.$$

**c.** Use the above identities to prove the Theorem about $\frac{1}{\exp_q(x)}$.

**d.** Determine the Mobius function $\mu[U, V]$ for any $U \subset V$ in $\mathcal{B}(q)$.