The Grassmannian $\mathrm{Gr}(d, F^n)$ is the parameter space whose points correspond to d-dimensional subspaces V in the n-dimensional vector space F^n over a given field F. We specify a subspace $V = \mathrm{Span}_F(v_1, \ldots, v_d)$ by a $d{\times}n$ matrix of row vectors, with change-of-basis symmetry group $\mathrm{GL}_d(F)$. This matrix can be normalized by making a given $d \times d$ submatrix into the identity, in columns $I = \{i_1 < \cdots < i_d\} \subset [n]$, provided the determinant of this submatrix is nonzero:

$$V \;=\; \mathrm{GL}_d \circlearrowright \begin{bmatrix} \text{---}\; v_1 \;\text{---} \\ \text{---}\; v_2 \;\text{---} \\ \vdots \\ \text{---}\; v_d \;\text{---} \end{bmatrix} \;=\; \begin{bmatrix} * * \cdots 1 \cdots 0 \cdots 0 \cdots * * \\ * * \cdots 0 \cdots 1 \cdots 0 \cdots * * \\ \vdots \\ * * \cdots 0 \cdots 0 \cdots 1 \cdots * * \end{bmatrix}$$

The $*$'s denote $d(n{-}d)$ free parameters in F defining a coordinate chart U_I of the Grassmannian, making it into an F-manifold: $\mathrm{Gr}(d, F^n) \;=\; \bigcup_I U_I$.

We define the Schubert cell decomposition $\mathrm{Gr}(d, F^n) \;=\; \coprod_I X_I$ by letting X_I consist of those $V \in U_I$ which have no $*$'s to the right of any 1 (row-echelon form). We can define X_I geometrically in terms of the standard basis $\{e_1, \ldots, e_n\}$ of F^n and the standard coordinate subspaces $E_r = \mathrm{Span}(e_1, \ldots, e_r)$; then $V \in X_I$ whenever $\dim(V{\cap}E_r) = \#(I{\cap}[r])$ for $r = 1, \ldots, n$. That is, $I = [d]$ forces $V = E_d$, and larger I makes $V \in X_I$ stick out further from the standard subspaces, until $I = \{n{-}d{+}1, \ldots, n\}$ corresponds to generic V's in the open set $X_I = U_I$. The topological closure $\overline{X}_I$ is given by: $\dim(V{\cap}E_r) \geqslant \#(I{\cap}[r])$ for $r = 1, \ldots, n$. We keep track of how the cells fit together using the Bruhat degeneration order: we define $I \leqslant J$ to mean $X_I \subset \overline{X}_J$, or equivalently $\overline{X}_I \subset \overline{X}_J$.

Example: For $\mathrm{Gr}(2, F^4)$, we have:

$$U_{34} \;=\; X_{34} \;=\; \begin{bmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix} \;=\; \{V \mid V{\cap}E_2 = 0\,, \dim(V{\cap}E_3) = 1\},$$

$$U_{14} \;=\; \begin{bmatrix} 1 & * & * & 0 \\ 0 & * & * & 1 \end{bmatrix}, \qquad X_{14} \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{bmatrix} \;=\; \{V \mid E_1 \subset V \not\subset E_3\}.$$

Here are the cell closures with defining conditions, height indicating Bruhat order:

$$\overline{X}_{34} = \mathrm{Gr}(2, F^4)$$

$$\overline{X}_{24} = (\dim(V{\cap}E_2){\geqslant}1)$$

$$\overline{X}_{23} = (V{\subset}E_3) \qquad\qquad \overline{X}_{14} = (E_1{\subset}V)$$

$$\overline{X}_{13} = (E_1{\subset}V{\subset}E_3)$$

$$\overline{X}_{12} = (V{=}E_2)$$

The Bruhat order relations $\overline{X}_I \subset \overline{X}_J$ are evident from the defining conditions on V. To verify in coordinates that $\{1, 3\} \leqslant \{1, 4\}$, we show that any plane $V_\circ \in X_{13}$ is approached by planes in X_{14}: we find a continuous family $\mathcal{V} : F \to \mathrm{Gr}(2, F^4)$ with $\mathcal{V}(t) \in X_{14}$ for $t \neq 0$, and $\mathcal{V}(0) = V_\circ$:

$$V_\circ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 \end{bmatrix}, \qquad \mathcal{V}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 1 & t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a/t & 1/t & 1 \end{bmatrix} \text{ for } t \neq 0.$$

Similarly, the flag manifold $\mathrm{Fl}(F^n)$ is the parameter space of flags

$$V_\bullet = (0 \subset V_1 \subset \cdots \subset V_{n-1} \subset F^n), \quad \dim(V_d) = d.$$

We specify $V_\bullet$ by a basis $\{v_1, \ldots, v_n\}$ of F^n, with $V_d = \mathrm{Span}(v_1, \ldots, v_d)$; the basis forms an $n{\times}n$ matrix of row vectors. The change-of-basis symmetry group B of $V_\bullet$ consists of all lower-triangular matrices (with non-zero diagonal entries) in $\mathrm{GL}_n(F)$, since we can add a multiple of v_i only to a later basis vector to leave each V_d invariant. We get a Schubert cell decomposition indexed by permutations $w \in S_n$: $\mathrm{Fl}(F^n) = \coprod_w X_w$, where X_w consists of $V_\bullet$ whose B-reduced form is a permutation matrix w, plus $*$ coordinates in the positions of the Röthe diagram $D(w) = \{(i, j) \mid j < w(i),\ i < w^{-1}(j)\}$. Thus $\dim(X_w) = \#D(w) = \mathrm{inv}(w)$.