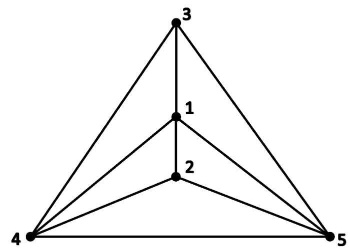
Homework: `math.msu.edu/~magyar/Math482/Old.htm#4-7`.



1a. In the graph above, each vertex v_i has two plane coordinates: $v_i = (x_i, y_i)$. We fix the outer vertices in some convenient way, for example $v_3 = (1,1),\ v_4 = (0,0),\ v_5 = (2,0)$. The degrees of freedom of our system are the four coordinates of the mobile inner vertices $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2)$.

1b. The forces are:

$$\begin{aligned}
F_1(x_1,y_1,x_2,y_2) &= (v_2-v_1)+(v_3-v_1)+(v_4-v_1)+(v_5-v_1)\\
&= (-4x_1{+}x_2{+}x_3{+}x_4{+}x_5,\ -4y_1{+}y_2{+}y_3{+}y_4{+}y_5),\\
&= (-4x_1{+}x_2{+}1{+}0{+}2,\ -4y_1{+}y_2{+}1{+}0{+}0),\\
F_2(x_1,y_1,x_2,y_2) &= (v_1-v_2)+(v_4-v_2)+(v_5-v_2)\\
&= (x_1{-}3x_2{+}x_4{+}x_5,\ y_1{-}3y_2{+}y_4{+}y_5).\\
&= (x_1{-}3x_2{+}0{+}2,\ y_1{-}3y_2{+}0{+}0).
\end{aligned}$$

To determine the correct sign of a term like $\pm(v_2 - v_1)$ in F_1, picture the physical vector in an example: if, as in the picture, $v_1 = (0.5, 0.6)$ and $v_2 = (0.5, 0.3)$, then $v_2 - v_1 = (0, -0.3)$, which pulls particle v_1 downward toward v_2, as desired. We only need to do this once: the remaining terms in F_1 will all be of the form $(v_i - v_1)$, and similarly for F_2.

1c. The equilibrium state is the value of (x_1, y_1, x_2, y_2) for which $F_1 = 0$ and $F_2 = 0$. That is, we must solve the linear system:

$$\begin{aligned}
-4x_1 + x_2 &= -3\\
-4y_1 + y_2 &= -1\\
x_1 - 3x_2 &= -2\\
y_1 - 3y_2 &= 0
\end{aligned}$$

This can be solved by Gaussian elimination, and I expect you to review this for the next Quiz. I just put it into Wolfram Alpha, which gave me (through its own Gaussian elimination):

$$v_1 = (x_1, y_1) = (1, \tfrac{3}{11}), \quad v_2 = (x_2, y_2) = (1, \tfrac{1}{11}).$$

Compare this to the centroid of the exterior triangle: $\frac{1}{3}(v_3 + v_4 + v_5) = (1, \frac{1}{3})$ is a bit below v_1, which looks reasonable.

1d. As in the Notes 4/7, we use the general formula for the potential in terms of the edges:

$$\begin{aligned} PE(x_1,y_1,x_2,y_2) &= \tfrac{1}{2}\sum_{ij\in E} |v_i - v_j|^2 = \tfrac{1}{2}\sum_{ij\in E} (x_i - x_j)^2 + (y_i - y_j)^2 \\ &= \frac{1}{2}\begin{pmatrix} (x_1-x_2)^2+(x_1-x_3)^2+(x_1-x_4)^2+(x_1-x_5)^2+(x_2-x_4)^2+(x_2-x_5)^2 \\ (y_1-y_2)^2+(y_1-y_3)^2+(y_1-y_4)^2+(y_1-y_5)^2+(y_2-y_4)^2+(y_2-y_5)^2 \end{pmatrix} + d, \end{aligned}$$

where $d = \frac{1}{2}(|v_3 - v_4|^2 + |v_3 - v_5|^2 + |v_4 - v_5|^2)$ is a constant. (As in the Notes, this can be obtained from a line integral in $\mathbb{R}^{10}$, moving all five vertices from the origin to their assigned positions.)

This means that the graph of PE is an upward-curving paraboloid, and its only critical point is a unique minimum.

1e. If any of the coordinates of v_1 or v_2 is large (positive or negative), then this vertex (say v_1) is far from the fixed vertices v_3, v_4, v_5. Taking a path from a fixed vertex to a far-away vertex, there must be at least one edge v_iv_j with a large distance

$$|v_i - v_j|^2 = (x_i - x_j)^2 + (y_i - y_j)^2,$$

which is one of the terms in the sums-of-squares forrmula for PE. Hence, as soon as any of x_1, y_1, x_2, y_2 is large positive or negative, $PE(x_1, y_1, x_2, y_2)$ has a large positive value. This guarantees that the diagonalized form of PE is a positive paraboloid, and has a unique critical point, its minimum.

2. All the computations are the same, except with 3 mobile vertices, with coordinates $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2)$, $v_3 = (x_3, y_3)$, and 3 force fields F_1, F_2, F_3. The equilibrium point will be found by solving a system of 6 linear equations in the 6 variables $x_1, \ldots, y_3$ (actually these can be separated into two systems in 3 variables each).

Extra credit if you can explicitly find the equilibrium positions of v_1, v_2, v_3 when the outer vertices are pinned in an equilateral triangle $v_4 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $v_5 = (1, 0)$, $v_6 = (0, 0)$. This might be the prettiest configuration for this graph.