Homework: math.msu.edu/∼magyar/Math482/Old.htm#4-21.

1. Easy from the definitions.

2. We have:

$$\sum_{n\geq 0} \Delta a_n x^n \;=\; \sum_{n\geq 0} a_n x^n - \sum_{n\geq 0} a_{n-1} x^n \;=\; f(x) - x f(x) \;=\; (1{-}x) f(x).$$

Similarly to this, we can derive the rules:

$$\begin{aligned}
\{a_n\}_{n\geq 0} &\stackrel{\text{ops}}{\longleftrightarrow} f(x)\\
\{\Delta a_n\}_{n\geq 0} &\stackrel{\text{ops}}{\longleftrightarrow} (1{-}x) f(x)\\
\{\Delta^+ a_n\}_{n\geq 0} &\stackrel{\text{ops}}{\longleftrightarrow} \tfrac{1}{x}\left((1{-}x)f(x) - a_0\right)\\
\{\Sigma a_n\}_{n\geq 0} &\stackrel{\text{ops}}{\longleftrightarrow} \tfrac{1}{1-x}\, f(x).
\end{aligned}$$

The third formula uses Wilf's Rule 1 (p. 34); the fourth formula is just Rule 5 (p. 37).

3. By translating the statement to generating function language, it becomes obvious:

$$\begin{aligned}
\{a_n\}_{n\geq 0} &\stackrel{\text{ops}}{\longleftrightarrow} f(x)\\
\{\Delta a_n\}_{n\geq 0} &\stackrel{\text{ops}}{\longleftrightarrow} (1{-}x) f(x)\\
\{\Sigma\Delta a_n\}_{n\geq 0} &\stackrel{\text{ops}}{\longleftrightarrow} \tfrac{1}{1-x}\,(1{-}x) f(x) = f(x)
\end{aligned}$$

Since the sequences $\{\Sigma\Delta a_n\}_{n\geq 0}$ and $\{a_n\}_{n\geq 0}$ have the same generating function, they must be equal sequences: $\Sigma\Delta a_n = a_n$ for all $n \geq 0$.

4. The equation $\Delta^+ a_n = a_n$ simply says $a_{n+1} - a_n = a_n$ or $a_{n+1} = 2a_n$, which has the obvious solution $a_n = a_0\, 2^n$, where a_0 is an arbitrary initial value.

Alternatively, translating to generating functions, the equation becomes:

$$\frac{(1{-}x)f(x) - a_0}{x} = f(x) \quad\Longleftrightarrow\quad f(x) = \frac{a_0}{1-2x} \quad\stackrel{\text{ops}}{\longleftrightarrow}\quad \{a_0\, 2^n\}_{n\geq 0}$$

This is closely analogous to the corresponding differential equation $a'(x) = a(x)$, which can be solved by separation of variables:

$$\tfrac{da}{dx} = a \quad\Leftrightarrow\quad \tfrac{da}{a} = dx \quad\Leftrightarrow\quad \int \tfrac{da}{a} = \int dx \quad\Leftrightarrow\quad \log(a) = x + c \quad\Leftrightarrow\quad a = e^{x+c} = a_0\, e^x$$

where $a_0 = a(0)$ is an arbitrary initial value. Thus we may say that the discete analog of $e = 2.71\ldots$ is just 2.

5. Easy from the definitions.

6. The difference equation $\Delta^+\Delta^- a_n = -a_n$ can be rewritten: $a_{n+1} - 2a_n + a_{n-1} = -a_n$ for $n \geq 1$, or $a_n = a_{n-1} - a_{n-2}$ for $n \geq 2$. This does not have any obvious solution, though it is clearly similar to the Fibonacci recurrence $F_n = F_{n-1} + F_{n-2}$. We will have to use generating functions.

Step 1: We must find a simple formula for $f(x) = \sum_{n\geq 0} a_n x^n$.

First Method. We use the recurrence to find an equation involving $f(x)$:

$$\begin{aligned}
f(x) &= a_0 + a_1 x + \textstyle\sum_{n\geq 2} a_n x^n\\
&= a_0 + a_1 x + \textstyle\sum_{n\geq 2} a_{n-1} x^n - \sum_{n\geq 2} a_{n-2} x^n\\
&= a_0 + a_1 x + x(f(x) - a_0) - x^2 f(x)
\end{aligned}$$

Solving this equation:

$$f(x) = \frac{a_0 - a_0 x + a_1 x}{1 - x + x^2}.$$

Second Method. Directly translate the difference equation into a generating function equation:

$$\Delta^+\Delta^- a_n = -a_n \text{ for } n \geq 1 \quad \overset{\text{ops}}{\longleftrightarrow} \quad \frac{(1-x)^2 f(x) - a_0}{x} - (-2a_0 + a_1) = -f(x) - (-a_0).$$

(The constant terms are subtracted from the generating functions because the sequence equation is not valid for $n = 0$. Also $\lim_{x \to 0} \frac{(1-x)^2 f(x) - a_0}{x} = -2a_0 + a_1$.) Solving, we get the same expression for $f(x)$.

Step 2. We must compute an explicit Taylor series for $f(x)$. As usual, we try to write the formula for $f(x)$ in terms of known series: in this case, a partial fracition decomposition into geometric series. Our answer will contain the arbitrary initial values a_0, a_1.

The roots of the denominator $1 - x + x^2$ are complex numbers[1] $\alpha = \frac{1+i\sqrt{3}}{2}$, $\beta = \frac{1-i\sqrt{3}}{2}$. We have: $1 - x + x^2 = (1 - x/\alpha)(1 - x/\beta)$. This is clear, because both sides have roots at $x = \alpha, \beta$, and the same constant coefficient 1.

The partial fraction decomposition must have the same vertical and horizontal asymptotes as $f(x)$:

$$f(x) \;=\; \frac{a_0 - a_0 x + a_1 x}{1 - x + x^2} \;=\; \frac{A}{1 - x/\alpha} + \frac{B}{1 - x/\beta}$$

Clearing denominators and simplifying $\frac{1}{\alpha} = \beta$ and $\frac{1}{\beta} = \alpha$ gives:

$$a_0 - a_0 x + a_1 x \;=\; A(1 - \tfrac{x}{\beta}) + B(1 - \tfrac{x}{\alpha}) \;=\; A(1 - \alpha x) + B(1 - \beta x).$$

Substituting $x = \alpha$ and $x = \beta$ gives:

$$A = \frac{a_0 - a_0\alpha + a_1\alpha}{1 - \alpha^2}, \qquad B = \frac{a_0 - a_0\beta + a_1\beta}{1 - \beta^2}.$$

Expanding the geometric series $\frac{A}{1 - x/\alpha} = \frac{A}{1 - \beta x} = \sum_{n \geq 0} A\beta^n$, and similarly for the other term, we conclude:

$$a_n = A\beta^n + B\alpha^n \quad \text{for } n \geq 0.$$

This might be simplified a bit by manipulating the expressions for A and B. The dependence on a_0, a_1 cannot be eliminated, since these are arbitrary constants. Of course, if a_0, a_1 are integers, then so are all the a_n's: the imaginary numbers and irrationals in the above formula all cancel out.

Analysis. As an example, consider $a_0 = 0$, $a_1 = 1$, so that:

n	0	1	2	3	4	5	6	7	8	9	10
a_n	0	1	1	0	-1	-1	0	1	1	0	-1

Now, we have $A = \frac{\alpha}{1-\alpha^2} = \frac{i2\sqrt{3}}{3}$, $B = \frac{\beta}{1-\beta^2} = -\frac{i2\sqrt{3}}{3}$, so

$$a_n = \frac{i2\sqrt{3}}{3}(\beta^n - \alpha^n)$$

Since $\alpha^6 = \beta^6 = 1$, this sequence has a period of 6: that is, $a_{n+6} = a_n$. This periodicity is analogous to the solutions of the corresponing continuous differential equation, the Hooke

[1] Actually, these numbers are reciprocals of each other, $\alpha\beta = 1$; and they are complex sixth roots of unity, $\alpha, \beta = \cos(\frac{2\pi}{6}) \pm i \sin(\frac{2\pi}{6})$, satisfying $\alpha^6 = \beta^6 = 1$. This is because $x^6 - 1 = (x+1)(x^2 - x + 1)(x^3 - 1)$.)

equation $a''(x) = -a(x)$. These are the wave functions $a(x) = a(0)\cos(x) + a'(0)\sin(x)$ with period 2π. Thus, we may say that the discrete analog of $\pi = 3.14\ldots$ is 3.

7. We seek a simple formula for $f(x) = \sum_{n\geq 0} n^{\underline{k}} x^n$. Consider $D^k(x^n) = n(n-1)\cdots(n-k+1)x^{n-k}$. Then $D^k \sum_{n\geq 0} x^n = \sum_{n\geq 0} n^{\underline{k}} x^{n-k}$, and:

$$f(x) \;=\; x^k D^k\left(\frac{1}{1-x}\right) \;=\; x^k D^k (1-x)^{-1} \;=\; x^k (1)(2)\cdots(k)(1-x)^{-k-1} \;=\; \frac{k!\, x^k}{(1-x)^{k+1}}.$$

8. PROPOSITION: $\Delta(n^{\underline{k}}) = k\,(n-1)^{\underline{k-1}}$.
Proof: Come to think of it, induction is not needed here. For any $n, k \geq 1$, we directly compute:

$$\begin{aligned}\Delta(n^{\underline{k}}) \;&=\; n(n-1)\cdots(n-k+1) \;-\; (n-1)\cdots(n-k+1)(n-k) \\ &=\; (n-(n-k))(n-1)^{\underline{k-1}} \;=\; k\,(n-1)^{\underline{k-1}}.\end{aligned}$$

9. The sequence $\{c_n\}_{n\geq 0}$ with $c_n = 0$ for $n < k$ and $c_k = 1$ for $n \geq k$ has generating function $\sum_{n\geq k} x^n = \frac{x^k}{1-x}$. Thus:

$$\sum_{n\geq 0} n^{\underline{k}} x^n = \sum_{n\geq 0} k!\,\Sigma^k(c_n) x^n = k!\left(\frac{1}{1-x}\right)^k \sum_{n\geq 0} c_n x^n = \frac{k!}{(1-x)^k}\cdot\frac{x^k}{1-x},$$

which is the same as in Prob. 7.

10. We seek a transformation proof for the formula: $n^k = \sum_{i=0}^k \left\{ {k \atop i} \right\} n^{\underline{i}}$.

The left side n^k is Twelvefold Way #1, counting all functions $f : [k] \to [n]$. On the right side, the Stirling partition number $\left\{ {k \atop i} \right\}$ is TW #9, counting the set partitions $\{S_1, \ldots, S_i\}$ where $S_1, \ldots, S_i$ are disjoint sets with $S_1 \cup \cdots \cup S_i = [k]$. Also, $n^{\underline{i}}$ is TW #2, counting injective functions $g : [i] \to [n]$.

We must transform the data of a function f into a pair: a set partition of $[k]$, and an injection g:

$$f \quad\longleftrightarrow\quad (S_1, \ldots, S_i\,; g) \;\text{ for some } i.$$

Given f, first define i to be the size of the output set of f (the image). Second, define the partition of $[k]$ by thinking of f as k labeled balls in n ordered bins; remove empty bins, and move the remaining bins into a standard order, so that $\min(S_1) < \min(S_2) < \cdots < \min(S_i)$, where $\min(S)$ means the smallest element of S. Third, define $g(j) = f(S_j)$, the common output of the elements in S_j.

The inverse transformation takes a pair $(S_1, \ldots, S_i; g)$, where $\min(S_1) < \cdots < \min(S_i)$, to the function f defined by $f(m) = g(j)$, for $m \in S_j$.

EXAMPLE: Consider the function $f : [7] \to [5]$ described either by a list of outputs $f = (f(1), \ldots, f(7))$, or 7 marked balls in 5 ordered bins:

$$f = (4, 1, 1, 5, 4, 1, 5) = 236|\,|\,|15|47.$$

First, we set $i = 3$, since f has 3 outputs 1,4,5. Second, we get a set partition by dropping empty baskets, considering the remaining baskets as unordered, exchangeable, and putting the baskets in standard order based on their minimal elements:

$$236|15|47 = 15|236|47.$$

The injective function $g : [3] \to [5]$ is $g(1) = f(1) = f(5) = 4$, and $g(2) = f(2) = f(3) = f(6) = 1$, and $g(3) = f(4) = f(7) = 5$. That is,

$$f = (3, 1, 1, 5, 3, 1, 5) = 236|\,|\,|15|47 \qquad\longleftrightarrow\qquad 15|236|47\;,\; g = (4, 1, 5).$$