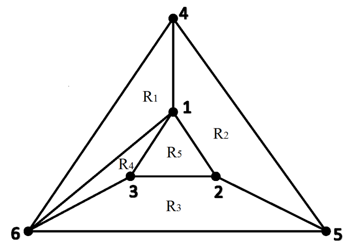
Homework: math.msu.edu/∼magyar/Math482/Old.htm#4-16.

1a. We have the chiral hexahedron graph G:



Given fixed external vertex positions:

$$v_4 = (1,1), \quad v_5 = (2,0), \quad v_6 = (0,0),$$

we wish to find the equilibrium positions of the mobile internal vertices v_1, v_2, v_3, where $v_i = (x_i, y_i)$. By HW 4/11, we must solve the matrix equations: $L \cdot \vec{x} = \vec{b}$ and $L \cdot \vec{y} = \vec{c}$. Here the partial Laplacian matrix L has rows and columns corresponding to the internal vertices v_1, v_2, v_3; diagonal entries are the degrees of these vertices; and there is an off-diagonal -1 for each edge $v_i v_j$:

$$L = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}.$$

The constant vectors $\vec{b} = (b_1, b_2, b_3)$, $\vec{c} = (c_1, c_2, c_3)$ are defined as b_i being the sum of x-coordinates of the fixed external neighbors of v_i, and similarly for c_i and y-coordinates: $\vec{b} = (1{+}0, 2, 0)$, $\vec{c} = (1{+}0, 0, 0)$. To solve these equations, we reduce the doubly-augmented matrix:

$$[\, L \mid \vec{b} \mid \vec{c}\,] \;=\; \left[\begin{array}{rrr|r|r} 4 & -1 & -1 & 1 & 1 \\ -1 & 3 & -1 & 2 & 0 \\ -1 & -1 & 3 & 0 & 0 \end{array}\right] \xrightarrow{\text{row red}} \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{13}{12} & \frac{1}{6} \\ 0 & 0 & 1 & \frac{7}{12} & \frac{1}{6} \end{array}\right].$$

Hence:

$$v_1 = (\tfrac{2}{3}, \tfrac{1}{3}), \quad v_2 = (\tfrac{13}{12}, \tfrac{1}{6}), \quad v_3 = (\tfrac{7}{12}, \tfrac{1}{6}).$$

Note that $v_3 v_1 v_4$ are collinear, and $v_3 v_2$ is a horizontal segment.

1b. We get a three-dimensional vector q_j for each of the 5 internal regions in the picture. For two adjacent regions $R_{\text{left}}, R_{\text{right}}$ separated by an edge with top and bottom vertices $v_{\text{top}} v_{\text{bot}}$, we find the vector for the right region from the known one for the left region, by the recursive rule:

$$q_{\text{right}} = q_{\text{left}} + (v_{\text{top}}, 1) \times (v_{\text{bot}}, 1)$$

Starting with $q_1 = (0,0,0)$, we obtain:

$$q_2 \;=\; q_1 + (v_4,1)\times(v_1,1) \;=\; (1,1,1)\times(\tfrac{2}{3},\tfrac{1}{3},1) \;=\; (\tfrac{2}{3},-\tfrac{1}{3},-\tfrac{1}{3})$$

$$q_3 \;=\; q_2 + (v_5,1)\times(v_2,1) \;=\; q_2 + (2,0,1)\times(\tfrac{13}{12},\tfrac{1}{6},1) \;=\; (\tfrac{2}{3},-\tfrac{1}{3},-\tfrac{1}{3}) + (-\tfrac{1}{6},-\tfrac{11}{12},\tfrac{1}{3}) \;=\; (\tfrac{1}{2},-\tfrac{5}{4},0)$$

$$q_4 \;=\; q_1 + (v_1,1)\times(v_6,1) \;=\; (\tfrac{2}{3},\tfrac{1}{3},1)\times(0,0,1) \;=\; (\tfrac{1}{3},-\tfrac{2}{3},0)$$

$$q_5 \;=\; q_4 + (v_1,1)\times(v_3,1) \;=\; q_4 + (\tfrac{2}{3},\tfrac{1}{3},1)\times(\tfrac{7}{12},\tfrac{1}{6},1) \;=\; (\tfrac{1}{3},-\tfrac{2}{3},0) + (\tfrac{1}{6},-\tfrac{1}{12},-\tfrac{1}{12}) \;=\; (\tfrac{1}{2},-\tfrac{3}{4},-\tfrac{1}{12}).$$

As a check, we alternatively compute:

$$q_5^{\text{alt}} \;=\; q_3 + (v_3,1)\times(v_2,1) \;=\; q_3 + (\tfrac{7}{12},\tfrac{1}{6},1)\times(\tfrac{13}{12},\tfrac{1}{6},1) \;=\; (\tfrac{1}{2},-\tfrac{5}{4},0) + (0,\tfrac{1}{2},-\tfrac{1}{12}) \;=\; (\tfrac{1}{2},-\tfrac{3}{4},-\tfrac{1}{12}).$$

Recall *Lemma 1:* We must have $q_5 = q_5^{\text{alt}}$. This is because the difference of the two sides corresponds to a sum of vector increments $(v_{\text{top}},1)\times(v_{\text{bot}},1)$ stepping around a cycle of neighboring regions, and this can can be rearranged as a sum of vector increments for the edges around vertices v_1 and v_2:

$$\begin{aligned} q_5 - q_5^{\text{alt}} \;&=\; (v_1,1)\times(v_6,1) \,+\, (v_1,1)\times(v_3,1) - (v_3,1)\times(v_2,1) - (v_5,1)\times(v_2,1) - (v_4,1)\times(v_1,1) \\ &=\; [(v_1,1)\times(v_6,1) \,+\, (v_1,1)\times(v_3,1) \,+\, (v_1,1)\times(v_2,1) \,+\, (v_1,1)\times(v_4,1)] \\ &\qquad + [(v_2,1)\times(v_1,1) \,+\, (v_2,1)\times(v_3,1) \,+\, (v_2,1)\times(v_5,1)] \end{aligned}$$

But now, from the first sum around v_1, we factor $(v_1,1)$, leaving the other factor: $(v_6,1) + (v_3,1) + (v_2,1) + (v_4,1) = 4(v_1,1)$ by the equilibrium condition; and $(v_1,1)\times 4(v_1,1) = (0,0,0)$. Similarly, the second sum around v_2 is also $(0,0,0)$.

1c. For v_i a corner of region R_j, we associate the height $h_i = q_j\cdot(v_i,1)$, which lifts the plane vector $v_i = (x_i,y_i)$ to the space vector $(v_i,h_i) = (x_i,y_i,h_i)$. We have:

$$h_1 = q_1\cdot(v_1,1) = 0, \quad h_2 = q_3\cdot(v_2,1) = \tfrac{1}{3}, \quad h_3 = q_3\cdot(v_3,1) = \tfrac{1}{12},$$

$$h_4 = q_1\cdot(v_4,1) = 0, \quad h_5 = q_3\cdot(v_5,1) = 1, \quad h_6 = q_1\cdot(v_6,1) = 0.$$

Therefore the vertices of P are:

$$v_1' = (\tfrac{2}{3},\tfrac{1}{3},0), \qquad v_2' = (\tfrac{13}{12},\tfrac{1}{6},\tfrac{1}{3}), \qquad v_3' = (\tfrac{7}{12},\tfrac{1}{6},\tfrac{1}{12}),$$

$$v_4' = (1,1,0), \qquad v_5' = (2,0,1), \qquad v_6' = (0,0,0).$$

1d. Each point $v\in R_j$ (not just corner vertices) gets a height $h(v) = q_j\cdot(v,1)$, so that v corresponds to a point $v' = (v,h(v))$ on a face of the polyhedron P. Letting $q_j = (a_j,b_j,c_j)$, we see that $v' = (x,y,z)$ satisfies the equation:

$$z \;=\; h(v) = a_j x + b_j y + c_j.$$

We can rewrite this as a vector equation defining the face of P corresponding to R_j:

$$-a_j x - b_j y + z = c_j \quad\iff\quad (-a_j,-b_j,1)\cdot(x,y,z) = c_j\,,$$

This is $q_j'\cdot v' = c_j$ for $q_j' = (-a_j,-b_j,1)$.

1e. Here is a table of q'_j heights for the vertices of P:

	v'_1	v'_2	v'_3	v'_4	v'_5	v'_6
q'_1	0	$\frac{1}{3}$	$\frac{1}{12}$	0	1	0
q'_2	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{3}$	0
q'_3	$\frac{1}{12}$	0	0	$\frac{3}{4}$	0	0
q'_4	0	$\frac{1}{12}$	0	$\frac{1}{3}$	$\frac{1}{3}$	0
q'_5	$-\frac{1}{12}$	$-\frac{1}{12}$	$-\frac{1}{12}$	$\frac{1}{4}$	0	0

For each region R_j, the vertices at its corners have the same q'_j-height, while the other vertices have greater q'_j-height. For example, the face above R_2 has its corners v'_1, v'_2, v'_4, v'_5 all at q'_2-height $-\frac{1}{3}$, whereas the other vertices v'_3, v'_6 have the greater heights $-\frac{1}{4}$ and 0.

This shows that the polyhedron P is defined by the conditions $q'_j \cdot (x, y, z) \geq d_j$ for $j = 1, \ldots 5$, along with one more condition coming from the top triangle (lid) with corners v'_4, v'_5, v'_6: this corresponds to an inequality $q'_0 \cdot (x, y, z) \leq d_0$ by HW 3/21 #2.

2. *Lemma 2:* If v is a corner of two regions S and R, then the heights associated to v by the vectors q_S and q_R are the same.

Proof: Once we know this for adjacent regions S, R, we can make a chain of equalities to connect any two regions around v.

Thus, it is enough to prove the Lemma in the case that S, R are neighbors across the boundary edge uv. Then $q_R = q_S + (v, 1)\times(u, 1)$, so that the two heights associated to v are $h_S(v) = q_S \cdot (v, 1)$, and:

$$h_R(v) \;=\; q_R \cdot (v, 1) \;=\; q_S \cdot (v, 1) \;+\; ((v, 1)\times(u, 1)) \cdot (v, 1).$$

But a basic property of the cross product is that $(v, 1)\times(u, 1)$ is orthogonal to $(v, 1)$ so the dot product on the very right of the equality is zero. The remaining term is precisely $h_S(v)$, as desired.