Homework: math.msu.edu/∼magyar/Math482/Old.htm#3-21.

1a. For $\vec{a} = (a_1, a_2)$, $\vec{x} = (x_1, x_2)$, we define the dot product as: $\vec{a} \cdot \vec{x} = a_1 x_1 + a_2 x_2$. The equation $\vec{a} \cdot \vec{x} = c$ means geometrically that the vector $\vec{x}$ lies on a line perpendicular to $\vec{a}$, at height c along $\vec{a}$. Our problem deals with three lines:

$$\begin{aligned} (1,1) \cdot (x_1, x_2) &= 2 \\ (-1,1) \cdot (x_1, x_2) &= 2 \\ (-1,2) \cdot (x_1, x_2) &= 0. \end{aligned}$$

Intersecting the first and third of these means solving the system of simultaneous linear equations:

$$\begin{aligned} x_1 + \ x_2 &= 2 \\ -x_1 + 2x_2 &= 0. \end{aligned}$$

This can be solved by Gaussian Elimination; or by multiplying with the inverse matrix; or by Cramer's Rule; or just by entering into Wolfram Alpha: `solve {x+y=2, -x+2y=0}`. Any such system will have a unique intersection point, or no intersection if the lines are parallel, or a whole line's worth of intersection if the lines are the same.

In our case, there is a unique intersection: $\vec{v}_1 = (x_1, x_2) = (\frac{4}{3}, \frac{2}{3})$, which is one vertex of the triangle cut out by the 3 lines. Intersecting the other two pairs of lines gives the other two vertices: $\vec{v}_2 = (0, 2)$ and $\vec{v}_3 = (-4, -2)$. Graph these vertices and lines for yourself to understand the geometric meaning.

1b. The inequality $\vec{a} \cdot \vec{x} \le c$ means geometrically that the point x lies on or below the line $\vec{a} \cdot \vec{x} = c$, and $\vec{a} \cdot \vec{x} \ge c$ means it lies on or above. That is, each inequality defines the half-plane below or above the line. (To be precise, *above* means on the same side as $\vec{a}$, *below* means on the opposite side.)

Two simultaneous inequalities define one of the angles between two lines: there are 4 such angles, corresponding to the 2^2 possible choices of $\ge$ or $\le$ in the two inequalities.

The three lines in part (a) cut out 1 triangle and 6 unbounded regions. We can graph these regions, and test one point in each to tell which inequalties define it.

For example, consider the centroid point $\vec{v}_0 = \frac{1}{3}\vec{v}_1 + \frac{1}{3}\vec{v}_2 + \frac{1}{3}\vec{v}_3 = (-\frac{8}{9}, \frac{2}{9})$. This point certainly lies inside the triangle region, so we can test: $(1,1) \cdot \vec{v}_0 = -\frac{2}{3} < 2$; that is, the triangle lies below the line $(1,1) \cdot \vec{x} = 2$. Testing the other lines, we find the triangle is given by the simultaneous inequalities:

$$\begin{aligned} x_1 + x_2 &\le 2 \\ -x_1 + x_2 &\le 2 \\ -x_1 + 2x_2 &\ge 0. \end{aligned}$$

The 6 unbounded regions correspond to different choices of the three inequalities: instead of $(\le, \le, \ge)$, we choose:

$$(\le, \le, \le),\ (\le, \ge, \le),\ (\ge, \le, \le),\ (\le, \ge, \ge),\ (\ge, \le, \ge),\ (\ge, \ge, \ge).$$

In fact, there is one unbounded region beyond each vertex and edge of the triangle, for a total of $3 + 3 = 6$.

1c. There are $2^3 = 8$ possible systems of 3 equalities, and we have accounted for only 7. What about the remaining one, $(\geq, \geq, \leq)$, which is the opposite of the inequlaties for the triangle? These are inconsistent inequalities, defining three half-spaces with no simultaneous intersection points. That is, they define the empty set.

1d. In 3-dimensional space $\mathbb{R}^3$, an equation $\vec{a} \cdot \vec{x} = c$ cuts out a plane perpendicular to $\vec{a}$, and an inequality $\vec{a} \cdot \vec{x} \leq c$ defines the half-space below the plane (i.e., on the opposite side from $\vec{a}$).

Doing the same exercise as (a)–(c) above with four vectors $\vec{a}_1, \ldots, \vec{a}_4 \in \mathbb{R}^3$ would define one bounded region, a tetrahedron (not a regular one, of course), and several unbounded regions. In fact, there is one unbounded region beyond each vertex, edge, and face of the tetrahedron, making $4 + 6 + 4 = 14$ unbounded regions. Again, this accounts for 15 of the $2^4 = 16$ possible choices for the four inequalities. The remaining choice is the one opposite to the inequalities for the tetrahedron; it is inconsistent, with empty solution set.

2a. We wish to find the inequalities defining the tetrahedron whose four vertices are:

$$\vec{v}_1{=}(1,1,1), \quad \vec{v}_2{=}(1,2,3), \quad \vec{v}_3{=}(-1,1,1), \quad \vec{v}_4{=}(0,0,0).$$

We take these three at a time, and find the corresponding plane H. Taking $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, two directions along the plane are: $\vec{v}_2 - \vec{v}_1$ and $\vec{v}_3 - \vec{v}_1$; and a normal vector (perpendicular to the plane) is given by their cross product:

$$\vec{a} = (\vec{v}_2 - \vec{v}_1) \times (\vec{v}_3 - \vec{v}_1).$$

Recall that the cross product of two vectors $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ is:

$$\vec{x} \times \vec{y} = \left(\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}, \; -\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}, \; \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right),$$

where we use the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

In our case, $\vec{x} = \vec{v}_2 - \vec{v}_1 = (0, 1, 2)$ and $\vec{y} = \vec{v}_3 - \vec{v}_1 = (-2, 0, 0)$; the normal vector is:

$$\vec{a} = (0, 1, 2) \times (-2, 0, 0) = (0, -4, 2).$$

Thus, the plane H containing the endpoints of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is defined by $\vec{a} \cdot \vec{x} = c$ for some c. To find c, we test the height of any of the three points of H, in direction $\vec{a}$:

$$c = \vec{a} \cdot \vec{v}_1 = (0, -4, 2) \cdot (1, 1, 1) = -2.$$

We conclude:

$$H \;=\; \left\{ \vec{x} \in \mathbb{R}^3 \text{ with } \vec{(0,-4,2)} \cdot \vec{x} = -2 \right\} \;=\; \{(x_1, x_2, x_3) \text{ with } -4x_2 + 2x_3 = -2\}$$

Now, the points of the tetrahedron are all above or below this plane. To find the correct inequality, I test the height of the remaining point $\vec{v}_4 = (0, 0, 0)$: that is, $\vec{a} \cdot \vec{v}_4 = (0,-4,2) \cdot (0,0,0) = 0 > c = -2$, so the points $\vec{x}$ of the tetrahedron are *above* the plane: $\vec{a} \cdot \vec{x} \geq c$.

Finding the other planes similarly, I get the following inequalities defining the tetrahedron (each corresponding to a plane H containing three of the points):

H_{123}	H_{124}	H_{134}	H_{234}
$(0,-4,2)\cdot\vec{x} \geq -2$	$(1,-2,1)\cdot\vec{x} \leq 0$	$(0,-2,2)\cdot\vec{x} \geq 0$	$(-1,-4,3)\cdot\vec{x} \leq 0.$

Note that the last 3 planes contain the origin $\vec{v}_4 = (0, 0, 0)$, so all have height $c = 0$.

2b. The centroid, or center of gravity, of the tetrahedron is the average of the four vertex vectors:

$$\tfrac{1}{4}(\vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4) = (\tfrac{1}{4}, 1, \tfrac{5}{4}).$$

2c. We wish to slice off a small piece of the corner near $\vec{v}_2 = (1,2,3)$. We will slice parallel to the plane opposite $\vec{v}_2$, namely $H = H_{134}$, which is defined by $\vec{a} \cdot \vec{x} = c$ for $\vec{a} = (0,-2,2)$. Now, the height of H is $c = 0$, while the height of $\vec{v}_2$ is $\vec{a} \cdot \vec{v}_2 = 2$; so we slice at a height above 0, and a bit below 2, say $\frac{3}{2}$. That is, our new polyhedron is below the slice hyperplane H_5, and corresponds to the inequality:

$$H_5 \ : \ (0,-2,2) \cdot \vec{x} \leq \tfrac{3}{2}.$$

This polyhedron is a *triangular prism*, but distorted, with top smaller than its base.

2d. The triangular prism in (d) above has the same base vertices v_1, v_3, v_4 as the tetrahedron, but instead of the top vertex $v_2 = (1,2,3)$, it has three new top vertices v_1', v_3', v_4', each given by intersections of three planes: triple intersections:

$$v_1' = H_5 \cap H_{123} \cap H_{124}\,, \qquad v_3' = H_5 \cap H_{123} \cap H_{234}\,, \qquad v_4' = H_5 \cap H_{124} \cap H_{234}\,.$$

Finding the intersection point $\vec{x} = (x_1, x_2, x_3)$ of three planes means solving a system of three simultaneous linear equations in the three variables x_1, x_2, x_3. For example, $\vec{x} = \vec{v}_1'$ is the solution of:

$$\begin{array}{rcrcr} H_5 & : & -2x_2 + 2x_3 & = & \frac{3}{2} \\ H_{123} & : & -4x_2 + 2x_3 & = & -2 \\ H_{124} & : & x_1 - 2x_2 + x_3 & = & 0 \end{array}$$

I get: $v_1' = (1, \frac{7}{4}, \frac{5}{2})$. You can find the other two corners similarly.

2e. Starting with the tetrahedron from (a), we add one new vertex v_5 slightly beyond the face H_{124}. The inequality defined by this face is: $(1,-2,1) \cdot \vec{x} \leq 0$. The center of the face is $\vec{v}_{124} = \frac{1}{3}(v_1 + v_2 + v_4) = (\frac{2}{3}, 1, \frac{4}{3})$. Thus, we take v_5 to be slightly above $\vec{v}_{124}$ in the normal direction of H_{124}:

$$v_5 = (\tfrac{2}{3}, 1, \tfrac{4}{3}) + \tfrac{1}{3}(1,-2,1) = (1, \tfrac{1}{2}, \tfrac{5}{3}).$$

This polyhedron is a *bipyramid*, since it is constructed by gluing together two (non-identical) triangular pyramids.