Homework: math.msu.edu/∼magyar/Math482/Old.htm#3-14.

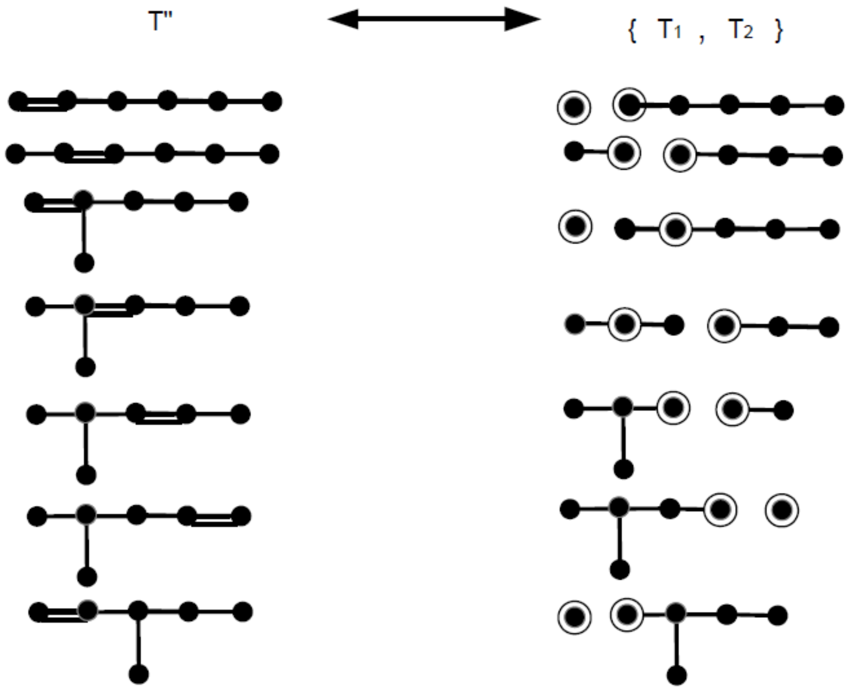
1a,b. The formula $q_T = p_T - 1$ is evident from the table in Solutions 3/12, #1. From the same table, $r_6 = 20$ (which agrees with the table in Solutions 3/10) and $e_6 = 14$, with the difference $t_6 = 20 - 14 = 6$, which is also in the table.

1c. The formula:

$$e_6 = \tfrac{1}{2}(r_1r_5 + r_2r_4 + r_3r_3 + r_4r_2 + r_5r_1 - r_3) = r_1r_5 + r_2r_4 + \tfrac{1}{2}r_3(r_3 - 1)$$

is checked using the table in Solutions 3/10 : $14 = (1)(9) + (1)(4) + \frac{1}{2}(2)(2-1)$.

1d. Here is half the table (which is all I had the patience to draw):



The full table would list on the left all 14 edge-rooted 6-vertex trees (with the root on a non-symmetry edge); and on the right all unordered pairs of *distinct* vertex-rooted trees with a total of 6 vertices.

3. One example of a graph with all of its edges being symmetry-edges is the cycle C_n, but there are many other highly symmetric graphs with this property. Another example is the edge-graph of the prism (cylinder) over an n-gon; this is the union of two n-cycles, one inside the other, with n extra edges joining corresponding pairs of vertices.

4a. An example of an infinite tree with infintely many symmetry edges is the infinite path with vertices v_i for all integers i, and edges v_iv_{i+1} for all i.

4b. An example of an infinite tree with just one symmetry edge is constructed from the infinite path above by adding "decorations" which limit the symmetry: for example, add two vertices x_0, x_1 and two edges v_0x_0 and v_1x_1. Then the only non-trivial symmetry of T is the reflection $\sigma(v_i) = v_{1-i}$, $\sigma(x_0) = x_1$, $\sigma(x_1) = x_0$; and the only symmetry edge is v_0v_1.

4c. Proposition: No tree T, finite or infinite, can have exactly 2 symmetry edges.

Proof: For reference in the following arguments: paths, walks, connected components, and cycles are defined in Graph Notes I and [HHM] pp. 6–8; trees are discussed in Graph Notes II.6 and [HHM] pp. 34–37.

Let T be a tree with symmetry edge $e = xy$, where $y = \sigma(x)$ for $\sigma \in \mathrm{Sym}(T)$.

Claim (i): If $x \neq \sigma(y)$, then T has infinitely many symmetry edges. Indeed, since $xy = x\,\sigma(x)$ is an edge of T, and σ takes edges to edges, we see $\sigma(x)\sigma(y) = \sigma(x)\sigma(\sigma(x))$ is also an edge. In fact, it is a symmetry edge, since the second vertex is σ of the first vertex. Applying σ repeatedly, we obtain an infinite walk of symmetry edges of the form $\sigma^k(x)\,\sigma^{k+1}(x)$ for all $k \geq 0$. Furthermore, $\sigma(\sigma(x)) = \sigma(y) \neq x$ by assumption, so this walk contains at least 3 distinct vertices $x, \sigma(x), \sigma^2(x)$. Now, if there were eventually a repeated vertex on the walk, it would create a cycle, which is impossible since T is a tree. Therefore, there are no repeated vertices, and we get an infinite path of symmetry edges in T.

Claim (ii): Removing $e = xy$ (but leaving its end vertices x, y) disconnects T into two connected components:

$$T{-}e \;=\; T_x \cup T_y.$$

First, x and y cannot be connected by a path P in $T{-}e$, since this would create a cycle $C = P{+}e$ in T, which is impossible. Second, since T is connected, every vertex v has a path $vw\cdots x$ in T. If the path does not contain e, then it is a path in $T{-}e$ from v to x. On the other hand, if the path does contain e, then it must be of the form $vw\cdots yx$, so $vw\cdots y$ is a path in $T{-}e$ from v to y. That is, every vertex is connected in $T{-}e$ to either x or y.

Claim (iii): If $x = \sigma(y)$, then $\sigma(T_x) = T_y$ and $\sigma(T_y) = T_x$. Since $\sigma(e) = \sigma(x)\sigma(y) = yx = e$, we see that σ takes paths in $T{-}e$ to other paths in $T{-}e$. Thus, σ takes the connected component of x to the connected component of $\sigma(x) = y$, and vice versa.

Claim (iv): If $e' = x'y'$ is another symmetry edge, then $e'' = \sigma(e')$ is also a symmetry edge. Indeed, it is an edge, because σ takes edges to edges. It is a symmetry edge, because for the composite symmetry $\tau = \sigma\sigma'\sigma^{-1}$, we have:

$$\begin{array}{rcll} \tau(x'') & = & \sigma(\sigma'(\sigma^{-1}(\sigma(x')))) & \text{since } \tau = \sigma\sigma'\sigma^{-1} \text{ and } x'' = \sigma(x') \\ & = & \sigma(\sigma'(x')) & \text{since } \sigma^{-1}\sigma = \epsilon, \text{ the identity symmetry} \\ & = & \sigma(y') & \text{since } \sigma'(x') = y' \\ & = & y'' & \text{since } y'' = \sigma'(y') \end{array}$$

We can now complete the proof. If $e' \subset T_x$, then e and $e' \subset T_x$ and $e'' \subset T_y$ are three distinct symmetry edges, and there cannot be only two; and similarly if $e' \subset T_y$.

Note: You can quote any of the above Claims in your own Hand-In proof.