**Inversion of Formal Power Series.** We extend the ring of formal power series $\mathbb{C}[\![x]\!]$ to the field of formal Laurent series $\mathbb{C}(\!(x)\!)$:

$$\mathbb{C}(\!(x)\!) = \left\{ \sum_{n \geq -N} a_n x^n \;\middle|\; N \in \mathbb{Z}, a_j \in \mathbb{C} \right\}.$$

These are the series in $x$ and $x^{-1}$ with a lowest term $x^{-N}$, but not necessarily a highest term. We define the operator $[x^n]$ which extracts the $x^n$ coefficient of a series: $[x^n]\left(\sum_j a_j x^j\right) = a_n$. The coefficient of $x^{-1}$ is the *residue* of a Laurent series; $[x^{-1}]$ is the *residue operator.*

LEMMA: (i) For $h(x) \in \mathbb{C}(\!(x)\!)$, we have $[x^{-1}]h'(x) = 0$.
(ii) For $h(x) \in x\mathbb{C}[\![x]\!]$ with $[x^1]h(x) \neq 0$, and $j \in \mathbb{Z}$, we have:

$$[x^{-1}]h(x)^j h'(x) = \begin{cases} 1 & \text{if } j = -1 \\ 0 & \text{else.} \end{cases}$$

That is, the residue of $h(x)^j h'(x)$ is always zero, except for $h(x)^{-1}h'(x)$.
*Proof.* (i) Obvious from the definition of derivative: $(x^n)' = nx^{n-1}$ for $n \in \mathbb{Z}$.
(ii) For $j \neq -1$, this follows from (i), since $h(x)^j h'(x) = \frac{1}{j+1}(h(x)^{j+1})'$.
For $j = -1$ and $h(x) = \sum_{n\geq 1} c_n x^n$ with $c_1 \neq 0$:

$$\begin{aligned} \frac{h'(x)}{h(x)} &= \frac{c_1 + 2c_2x + \cdots}{c_1x + c_2x^2 + + \cdots} = \frac{c_1 + 2c_2x + \cdots}{c_1x} \cdot \frac{1}{1 + \left(\frac{c_2}{c_1}x + \frac{c_3}{c_1}x^2 + \cdots\right)} \\ &= \left(x^{-1} + \tfrac{2c_2}{c_1} + \tfrac{3c_3}{c_1}x + \cdots\right)\left(1 - x\left(\tfrac{c_2}{c_1} + \tfrac{c_3}{c_1}x + \cdots\right) + \cdots\right), \end{aligned}$$

from which $[x^{-1}]\, h(x)^{-1}h'(x) = 1$ is evident. □

LAGRANGE INVERSION THEOREM: Let $f(x), g(x) \in x\mathbb{C}[\![x]\!]$ be inverses: $g(f(x)) = x$. Then:

$$[x^n]f(x) = \frac{1}{n}[x^{-1}]\frac{1}{g(x)^n}.$$

In particular, if $g(x) = x/\phi(x)$ and $f(x) = x\,\phi(f(x))$, then:

$$[x^n]f(x) = \frac{1}{n}[x^{n-1}]\phi(x)^n.$$

*Proof.* Letting $f(x) = \sum_{i\geq 1} a_j x^j$, our goal is to to compute a particular $a_n$ in terms of $g(x)$. Since $g = f^{-1}$, we have:

$$x = f(g(x)) = \sum_{j\geq 1} a_j g(x)^j.$$

We need to extract $a_n$ from this summation, making all other $a_j$'s vanish. We will manipulate the summation to get $a_n$ as the coefficient of $g(x)^{-1}g'(x)$, and all the rest of the $a_j$'s as coefficients of $g(x)^j g'(x)$; then apply the residue operator $[x^{-1}]$ in the Lemma. First, we take the derivative:

$$1 = \sum_{j\geq 1} ja_j \left(g(x)^j\right)' = \sum_{j\geq 1} ja_j\, g(x)^{j-1}g'(x).$$

Next, we move the $a_n$ term to be the coefficient of $g(x)^{-1}g'(x)$, dividing by $g(x)^n$:

$$\begin{aligned}\frac{1}{g(x)^n} &= \sum_{j\geq 1} ja_j\, g(x)^{j-1-n} g'(x) \\ &= \sum_{j=1}^{n-1} \frac{ja_j}{i-n}\left(g(x)^{j-n}\right)' \;+\; na_n\frac{g'(x)}{g(x)} \;+\; \sum_{j>n}\frac{ja_j}{j-n}\left(g(x)^{j-n}\right)'.\end{aligned}$$

Applying $[x^{-1}]$, we obtain essentially the first part of the Theorem: $[x^{-1}][1/g(x)^n] = na_n$.

For the second part, take $g(x) = x/\phi(x)$ so that $x = g(f(x)) = f(x)/\phi(f(x))$ is equivalent to $f(x) = x\,\phi(f(x))$. Now, evidently $[x^{-1}]h(x) = [x^{n-1}](x^n h(x))$, so:

$$a_n = \frac{1}{n}[x^{-1}]\frac{1}{g(x)^n} = \frac{1}{n}[x^{n-1}]\frac{x^n}{g(x)^n} = \frac{1}{n}[x^{n-1}]\frac{x^n}{x^n/\phi(x)^n} = \frac{1}{n}[x^{n-1}]\phi(x)^n. \qquad \square$$

Reference: Richard Stanley, *Enumerative Combinatorics*, Vol. 2, Ch. 5.

**Inversion of Analytic Functions.** We give an analytic derivation of Lagrange Inversion. Consider an analytic function $g(u)$ with $g(0) = 0$ and $g'(0) \neq 0$, so that by the Inverse Function Theorem, $g(u)$ is one-to-one inside a small circle $\mathcal{C}$ defined by $|u| = \delta$. Then $z = g(u)$ takes $\mathcal{C}$ to a simple closed curve $g(\mathcal{C})$ around $z = 0$, and there is a unique inverse function $f(z)$ defined inside $g(\mathcal{C})$.

Our aim is to compute the Taylor coefficients of $f(z) = \sum_{n\geq 0} a_n z^n$ in terms of its inverse function $g(u)$.

Recall the idea behind Cauchy's Integral Formula. In the complex plane, for any simple closed curve around $z = 0$, we have the line integral $\oint z^n\,dz = 0$ if $n \neq -1$, and $\oint z^{-1}\,dz = 2\pi i$. Hence, for any meromorphic function $h(z) = \sum_{n\geq -N} c_n z^n$, the residue is $c_1 = [z^{-1}]h(z) = \frac{1}{2\pi i}\oint h(z)\,dz$, and we can extract any coefficient of the Laurent series as: $c_n = \frac{1}{2\pi i}\oint h(z)/z^{n+1}\,dz$.

In our case, we have:

$$a_n = \frac{1}{2\pi i}\oint_{g(\mathcal{C})}\frac{f(z)}{z^{n+1}}\,dz.$$

Making the change of variable $z = g(u),\ u = f(z),\ dz = g'(u)\,du$:

$$a_n = \frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{u}{g(u)^{n+1}}\,g'(u)\,du = \frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{g'(u)}{g(u)^{n+1}}\,u\,du.$$

Performing integration by parts, we have: $\left(\frac{-1}{ng(u)^n}\right)' = \frac{g'(u)}{g(u)^{n+1}}$ and:

$$a_n = \left[\frac{-1}{ng(u)^n}\,u\right]_{u=\delta}^{u=\delta} - \frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{1}{n}\frac{-1}{g(u)^n}\,du = \frac{1}{n}\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{1}{g(u)^n}\,du.$$

The last integral is the residue:

$$a_n = \frac{1}{n}\,[u^{-1}]\frac{1}{g(u)^n}\,.$$

**Generalization.** We can use the same reasoning as above to find the Taylor coefficients of the composition $h(f(x))$ for any power series $h(x) \in x\mathbb{C}[\![x]\!]$:

$$[x^n]h(f(x)) = \frac{1}{n}\,[x^{-1}]\frac{h'(x)}{g(x)^n}.$$