Math 418H **Fall 2005**

Lecture: Fri 11/18

1. Group Theory: Algebra of Symmetry

   - Let X be a geometric object, a set of points with some geometric structure. A *symmetry* of X is a mapping of X onto itself, preserving the structure: $\pi : X \to X$.
   - Given two symmetries of X, we can compose them to get another symmetry: $\gamma = \alpha \bullet \beta$ means $\gamma : X \to X$ with $\gamma(x) := \alpha(\beta(x))$ for each point $x \in X$.
   - *Group:* $(G, \bullet)$ is the set $G = \mathrm{Sym}(X)$ of all symmetries of an object X, along with the composition operation $\bullet$.
   - *Example:* Let X be the human body. There are two symmetries: the identity mapping ι which takes each point to itself: $\iota(x) := x$; and the bilateral reflection σ which switches each point on the left with the corresponding point on the right. Flipping twice takes every point to itself, so $\sigma \bullet \sigma = \iota$. Further, ι is an identity element for this operation: $\iota \bullet \sigma = \sigma \bullet \iota = \sigma$.

2. Formal definition of a group

   - $(G, \bullet)$, where G is a set and $\bullet$ is a binary operation on G which satisfies the same axioms as multiplication in a ring.
   - associativity $(\alpha \bullet \beta) \bullet \gamma = \alpha \bullet (\beta \bullet \gamma)$
   - identity: there is an element ι with $\iota \bullet \alpha = \alpha \bullet \iota = \alpha$
   - inverses: for *every* α there is a $\beta = \alpha^{-1}$ with $\alpha \bullet \beta = \beta \bullet \alpha = \iota$
   - Clearly, $G = \mathrm{Sym}(X)$ with the composition $\bullet$ satisfies these axioms:
     $$(\alpha \bullet \beta) \bullet \gamma(x) = \alpha \bullet (\beta \bullet \gamma)(x) = \alpha(\beta(\gamma(x))) .$$
     Also, the identity symmetry ι is the group identity, and the inverse of a symmetry is the map which undoes it: $\alpha^{-1}(x) = y$, where $y = \alpha(x)$.
   - It is more difficult, but possible, to show that any group is the symmetry group of some object X (indeed, there are many such) .

3. Symmetric group $G = S_n$

- $X = \{1, 2, \ldots, n\}$, an unstructured set of n points. A symmetry is any permutation (a shuffling, or one-to-one correspondence) of these points.
- Denote permutations with the *two-line notation*: $\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}$
- *Example:* For $n = 3$, $X = \{1, 2, 3\}$ we have:

  $$S_3 = \left\{ \iota = \begin{pmatrix} 123 \\ 123 \end{pmatrix}, \begin{pmatrix} 123 \\ 132 \end{pmatrix}, \begin{pmatrix} 123 \\ 213 \end{pmatrix}, \begin{pmatrix} 123 \\ 231 \end{pmatrix}, \begin{pmatrix} 123 \\ 312 \end{pmatrix}, \begin{pmatrix} 123 \\ 321 \end{pmatrix} \right\}$$

  The permutation $\pi = \begin{pmatrix} 123 \\ 312 \end{pmatrix}$ means $\pi(1) = 3$, $\pi(2) = 1$, $\pi(3) = 2$.
- The group operation means doing one permutation after the other. E.g.:

  $$\alpha = \begin{pmatrix} 123 \\ 132 \end{pmatrix}, \ \beta = \begin{pmatrix} 123 \\ 231 \end{pmatrix}, \ \alpha \bullet \beta = \begin{pmatrix} 123 \\ 321 \end{pmatrix}, \ \beta \bullet \alpha = \begin{pmatrix} 123 \\ 213 \end{pmatrix}.$$

  since $\alpha(\beta(1)) = \alpha(2) = 3$, so $(\alpha \bullet \beta)(1) = 3$, etc.
- In general, the total number of permuations is $|S_n| = n!$, since we have n choices for $\pi(1)$, then $(n-1)$ different choices for $\pi(2)$, etc.

4. Symmetries of a triangle $G = D_3$

- X = an equilateral triangle, considered as a rigid object in the plane. A symmetry is a map $\alpha : X \to X$ which preserves the distance between points (say, a reflection or rotation of the triangle onto itself). We call this symmetry group $G = D_3$.
- Each corner must map to another corner under a symmetry. Labelling the corners by $\{1, 2, 3\}$, we can consider any symmetry as a permutation: $D_3 \subset S_3$. For example, the reflection which fixes 1 and switches 2,3 is the permutation $\begin{pmatrix} 123 \\ 132 \end{pmatrix}$.
- We can easily see that *every* permutation in S_3 corresponds to a symmetry of the triangle, so $D_3 = S_3$.
- *Exercise:* Write the 6×6 multiplication table of $D_3 = S_3$. It helps to denote each element by a letter (e.g., ι, α, β defined above).
- *Exercise:* Work all this out for D_4, the symmetries of a square. Note that *not* every permutation in S_4 corresponds to a symmetry of the square. Indeed, $|D_4| = 8$, but $|S_4| = 24$.