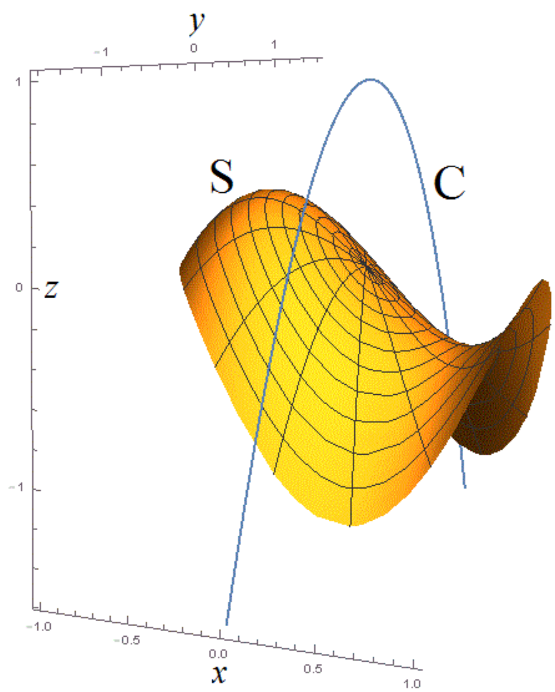
Math 254H **Weekly Homework 9** **Apr 17, 2017**

Consider the surface S which is the graph of the function $f(x,y) = x^3 - x - y^2$ over the unit disk $x^2+y^2 \leq 1$:

$$S = \{\, (x,y,z) \in \mathbb{R}^3 \;\mid\; z = x^3 - x - y^2, \quad x^2+y^2 \leq 1 \,\}.$$

Let C be the parabolic curve with $x = 0$, $z = 1 - y^2$.

PROBLEM: Find the minimum distance between S and C. That is, find the points $\vec{u} \in S$ and $\vec{v} \in C$ which minimize the function $\operatorname{dist}(\vec{u}, \vec{v}) = |\vec{u} - \vec{v}|$.



1. Explain in detail why the problem is equivalent to minimizing the function:

$$\ell(x,y,t) = x^2 + (y - t)^2 + (x^3 - x - y^2 - 1 + t^2)^2$$

over the solid cylindrical region $R = \{(x,y,t) \in \mathbb{R}^3 \mid x^2+y^2 \leq 1\}$.

2. Find the critical points $\vec{a}$ of $\ell(x,y,t)$ which lie in region R. (Any $\vec{a}$ outside of R are irrelevant.) Get approximate solutions to $\nabla\ell(\vec{a}) = \vec{0}$ using Wolfram Alpha or Mathematica. Determine $f(\vec{a})$ for each critical point.

Examples of Mathematica code: `h[x_,y_]:=x^2+xy; h[1,2]; D[h[x,y],x]; NSolve[{D[h[x,y],x], D[h[x,y],y]} == {0,0}].` Press Ctrl+Enter to evaluate an expression. Ending a line with ; means the output result is not printed, using no end-punctuation means it is printed.

3. For each critical point found, classify it as max, min, or indefinite (saddle), using the Second Derivative Test for the 3×3 Hessian $Hf_{\vec{a}}$ on [MT] p. 175. (If $Hf_{\vec{a}}$ is positive definite, $\vec{a}$ is a min; if $-Hf_{\vec{a}}$ is positive definite, $\vec{a}$ a max; otherwise, $\vec{a}$ is a saddle.) Note it is enough to test $Hf_{\vec{a}}$ at a numerical approximation of $\vec{a}$, since the sign of a determinant will not change because of a slight error in $\vec{a}$ (unless the det is zero).

4. Find the approximate critical min point of $\ell(x,y,t)$ using the gradient-flow method on a spreadsheet, from WHW 3.

5. Repeat #2 and/or #4 for the boundary surface of the region R. That is parametrize the surface $B = \{(x,y,t) \in \mathbb{R}^3 \mid x^2+y^2 = 1\}$, as $B(s,t) = (x(s), y(s), t)$, and find the critical points of $\ell^*(s,t) = \ell(B(s,t))$ over the (s,t) parameter region.

6. Again find the minimum points on B using the Lagrange multiplier method on [MT] p. 185. That is, to find critical points of $\ell(x,y,t)$ on the constraint surface $g(x,y,t) = x^2+y^2 = 1$, solve the simultaneous equations $g(x,y,t) = 1$ and $\vec{\nabla}\ell(x,y,t) = \lambda\vec{\nabla}g(x,y,t)$ for the four variables x,y,t,λ. This means the points on the surface where the gradient of ℓ is parallel to the gradient of the constraint function g, which implies that the level surface of ℓ is tangent to the constraint surface.

7. Finally, determine the overall minimum value of $\ell(x,y,t)$ on the region R, and solve the original problem. Does your answer seem right on the picture?