Math 254H **Homework 3 Solutions** **Spr 2017**

1a. Using the matrix for $P = \text{Proj}_{\mathbf{a}}$ in HW 2 Solutions, and assuming $|\mathbf{a}| = a_1^2 + a_2^2 = 1$, we compute:

$$[P] \cdot [P] \;=\; \begin{bmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{bmatrix} \cdot \begin{bmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{bmatrix} \;=\; \begin{bmatrix} a_1^4 + a_1^2 a_2^2 & -a_1 a_2^3 - a_1^3 a_2 \\ -a_1 a_2^3 - a_1^3 a_2 & a_1^2 a_2^2 + a_1^4 \end{bmatrix}$$

The identity $a_1^2 + a_2^2 = 1$ easily reduces this to the orignal matrx of P, agreeing with HW 2 #2a.

1b. Similar to #1a above.

1c. A special case off #3 below.

2. The matrix of a mapping $\ell : \mathbb{R}^n \to \mathbb{R}^m$ has m rows and n columns (dimensions $m \times n$). The given mappings have matrices

$$[\,\ell_1\,] \;=\; [\,m_1 \;\; m_2\,], \qquad [\,\ell_2\,] \;=\; \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

Two functions can be composed only if the output of the second function is an acceptable input of the first function; thus $\ell_1 \circ \ell_1$ and $\ell_2 \circ \ell_2$ are not defined.

The allowable compositions are $\ell_1 \circ \ell_2 : \mathbb{R} \to \mathbb{R}$, with $\ell_1(\ell_2(t)) = (m_1 a_1 + m_2 a_2)\, t = (\mathbf{m} \cdot \mathbf{a})\, t$, and:

$$[\,\ell_2 \circ \ell_1] \;=\; \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cdot [\,m_1 \;\; m_2\,] \;=\; \begin{bmatrix} a_1 m_1 & a_1 m_2 \\ a_2 m_1 & a_2 m_2 \end{bmatrix}.$$

This is not very nice geometrically, but it can be thought of as projecting onto the direction $\mathbf{m} = (m_1, m_2)$, dilating by a factor of $|\mathbf{a}|\,|\mathbf{m}|$, then rotating the direction $\mathbf{m}$ to the direction $\mathbf{a}$.

3. The relation $\text{Rot}_{\alpha+\beta} = \text{Rot}_\alpha \circ \text{Rot}_\beta$, translates into the matrix equations:

$$\begin{aligned}
\begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} \;&=\; [\text{Rot}_{\alpha+\beta}] \;=\; [\text{Rot}_\alpha \circ \text{Rot}_\beta] \;=\; [\text{Rot}_\alpha] \cdot [\text{Rot}_\beta] \\
&=\; \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix} \\
&=\; \begin{bmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) & -\cos(\alpha)\sin(\beta) - \sin(\alpha)\cos(\beta) \\ \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) & -\sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) \end{bmatrix}
\end{aligned}$$

Equating entries in the first and last matrices gives the Angle Addition Formulas.