Lect 3 I. Gradient & linear approximation
II. Reversing derivatives (dim 1)
III. Reversing derivatives (dim 2)
IV. Computing line integrals

<u>Quiz 3</u>: Gradient field & contours of $f(x,y) = \sqrt{x^2+y^2}$

Note: $|\vec{\nabla} f(x,y)| = \left| \frac{(x,y)}{\sqrt{x^2+y^2}} \right| = 1$ = uphill slope of graph

Contours are circles at even intervals $\perp \nabla f$

Graph $z = f(x,y)$ is <u>cone</u> flaring upward from $(0,0)$

I. Gradient & linear approximation

(dim 1) $f : \mathbb{R} \to \mathbb{R}$ near $x = a$

$$f(x) \approx \underbrace{f(a) + f'(a)(x-a)}_{\text{graph = tangent line}}$$

(affine approximation = constant + linear)

(dim 2) $f : \mathbb{R}^2 \to \mathbb{R}$ near $(x,y) = (a,b)$

$$f(x,y) \approx f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

graph = tangent plane

Vector notation
Let $\vec{v} = (x,y)$
$\vec{c} = (a,b)$

$$f(\vec{v}) \approx f(\vec{c}) + \underbrace{\vec{\nabla} f(a,b)}_{\substack{\text{vector of slopes}\\ \text{above } (a,b)}} \cdot (\vec{v} - \vec{c})$$

Recall: linear fcn $\vec{\ell}(\vec{v}) = \vec{m} \cdot \vec{v}$

contour lines orthogonal to $\vec{m}$ = uphill vector
(e.g. $\vec{m} \cdot \vec{v} = 0$ zero-contour)

So: $\vec{\nabla} f(\vec{c})$ is orthogonal to contour curve of f through $\vec{c}$: $\nabla f(\vec{c})$ = uphill vector

II. Reversing derivatives (dim 1)

$f : \mathbb{R} \to \mathbb{R}$, Given $f'(x) = g(x)$ known rate of change

Known initial value $f(0)$

Want to find original $f(x)$.

<u>Fundamental Theorem of Calculus:</u>

Total change equals integral of rate of change

$$f(a) - f(0) = \int_0^a f'(x)\,dx$$

$\left[\text{so } f(a) = f(0) + \int_0^a f'(x)\,dx \text{, reconstruct } f \right]$

Analyze <u>why</u>, so as to do the same in dim 2.

Mark interval $[0, a]$ with n sample points

$\Delta x_i = x_i - x_{i-1}$ increment (of x)

$\Delta f(x_i) = f(x_i) - f(x_{i-1})$ (of f)

Integral $\int_0^a f'(x)\,dx \approx \sum_{i=1}^n f'(x_i)\,\Delta x_i$

Have:

$$\boxed{f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \frac{\Delta f(x_i)}{\Delta x_i}}$$

$$\approx \sum_{i=1}^n \frac{\Delta f(x_i)}{\Delta x_i}\,\Delta x_i = \sum_{i=1}^n \Delta f(x_i)$$

$$= f(a) - f(0)$$

total change in f = sum of incremental changes in f

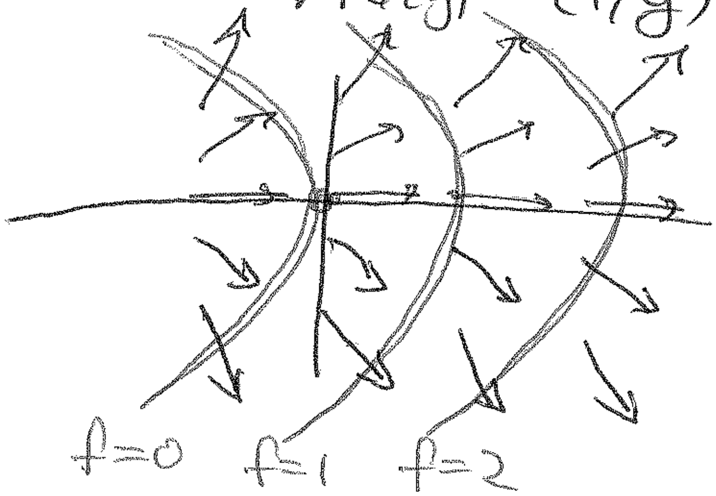
Approximations become exact as $n \to \infty$, $\Delta x_i \to 0$

Again: total change = sum of incremental changes

$$f(a)-f(0) = \Delta f(x_1) + \Delta f(x_2) + \cdots + \Delta f(x_n)$$
$$\approx f'(x_1)\Delta x_1 + f'(x_2)\Delta x_2 + \cdots + f'(x_n)\Delta x_n$$
$$\approx \int_0^a f'(x)\,dx \quad \text{with } \approx \text{ becoming } = \text{ as } n\to\infty$$

III. Reversing gradients (dim 2)

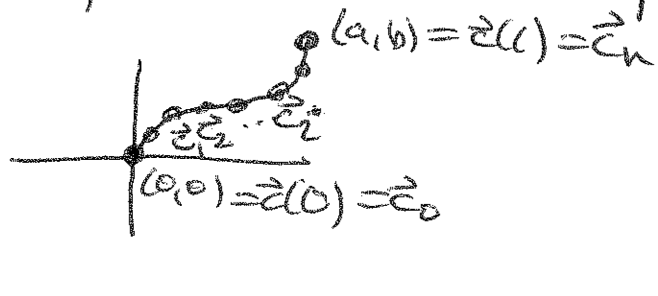
Example: Given $\vec{\nabla} f(x,y) = (1,y)$, $f(0,0)$, describe f.



draw contours orthogonal to $\vec{\nabla} f$

Graph of $z=f(x,y)$ = ascending parabolic trough

Qualitatively, it is clear how $\vec{\nabla} f$ determines f.
How to compute? Imitate strategy for dim 1.
To determine total change $f(a,b)-f(0,0)$,
draw a curve $\vec{c}(t)$ from $\vec{c}(0)=(0,0)$
to $\vec{c}(1)=(a,b)$, cut into n sample points
$\vec{c}_i = \vec{c}(t_i)$



total change = sum of incremental changes

$$f(a,b) - f(0,0) = \Delta f(\vec{c}_1) + \Delta f(\vec{c}_2) + \cdots + \Delta f(\vec{c}_n)$$

where $\Delta f(c_i) = f(\vec{c}_i) - f(\vec{c}_{i-1})$, incremental change

To find, use affine approx near $\vec{v} = \vec{c}_{i-1}$

$$f(\vec{v}) \approx \vec{\nabla} f(\vec{c}_{i-1}) \cdot (\vec{v} - \vec{c}_{i-1}) + f(\vec{c}_{i-1})$$

Then $f(\vec{c}_i) - f(\vec{c}_{i-1}) \approx \vec{\nabla} f(\vec{c}_{i-1}) \cdot (\vec{c}_i - \vec{c}_{i-1}) + \cancel{f(\vec{c}_{i-1})} - \cancel{f(\vec{c}_{i-1})}$

$$\Delta f(\vec{c}_i) \approx \vec{\nabla} f(\vec{c}_{i-1}) \cdot \Delta \vec{c}_i$$

Thus: $f(a,b) - f(0,0) \approx \sum_{i=1}^{n} \vec{\nabla} f(\vec{c}_i) \cdot \Delta \vec{c}_i$

Now: cook up a new kind of integral which is the limit of the right-hand side:

$$\oint_{\vec{c}} \vec{F}(\vec{c}) \cdot d\vec{c} \stackrel{def}{=} \lim_{n \to \infty} \sum_{i=1}^{n} \vec{F}(\vec{c}_i) \cdot \Delta \vec{c}_i$$

This defines "line integral of $\vec{F}$ (vector field) along curve $\vec{c}$"

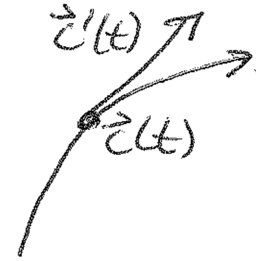
<u>Gradient Theorem:</u>

$$f(a,b) - f(0,0) = \oint_{\vec{c}} \vec{\nabla} f(\vec{c}) \cdot d\vec{c}$$

for any curve $\vec{c}$ from $(0,0)$ to (a,b)

We <u>defined</u> $\oint$ so that this would work.

IV. Computing line integrals



Curve $\vec{c}(t) = (x(t), y(t))$

Velocity vector $\vec{c}\,'(t) = (x'(t), y'(t))$

definition $$= \lim_{\Delta t \to 0} \frac{\vec{c}(t+\Delta t) - \vec{c}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta c(t)}{\Delta t}$$

For Δt small, $\Delta \vec{c}(t) \approx \vec{c}\,'(t)\Delta t$

$$\Delta \vec{c}_i = \vec{c}(t_i) - \vec{c}(t_{i-1}) \approx \vec{c}\,'(t_i)\,\Delta t$$

Thus $$\oint_{\vec{c}} \vec{F}(\vec{c}) \cdot d\vec{c} = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}(\vec{c}_i) \cdot \Delta \vec{c}_i$$

$\boxed{\vec{F}(x,y) = (p(x,y), q(x,y))}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}(\vec{c}(t_i)) \cdot \vec{c}\,'(t_i)\,\Delta t$$

$$= \int_0^1 \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$$

$$= \int_0^1 \vec{F}(x(t), y(t)) \cdot (x'(t), y'(t))\, dt$$

$$= \int_0^1 p(x(t), y(t))\, x'(t) + q(x(t), y(t))\, y'(t)\, dt$$

This is a (complicated) ordinary, dimension-1 integral.

<u>Ex:</u> $\nabla f(x,y) = \vec{F}(x,y) = (1, y)$, $f(x,y) = ?$

Take $\vec{c}(t) = (ta, tb)$ from $c(0) = (0,0)$ to $c(1) = (a,b)$

Then $f(a,b) = \cancel{f(0,0)}^{\,0} + \int_{\vec{c}} \vec{F}(\vec{c}) \cdot d\vec{c} = \int_0^1 \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$

$$= \int_0^1 \vec{F}(ta, tb) \cdot (ta, tb)'\, dt = \int_0^1 (1, tb) \cdot (a, b)\, dt$$

$$= \int_0^1 a + tb^2\, dt = ta + \tfrac{1}{2}t^2 b^2 \Big|_{t=0}^{t=1} = a + \tfrac{1}{2}b^2 \Rightarrow f(x,y) = x + \tfrac{1}{2}y^2$$