Math 254H **Limits and Differentiation** **Feb 5, 2016**

1 Notation. We denote a vector in n-dimensional space $\mathbb{R}^n$ by:

$$\vec{x} = (x_1, \ldots, x_n) = x_1\vec{e}_1 + \cdots + x_n\vec{e}_n,$$

where the standard basis vector $\vec{e}_i = (0, \ldots, 1, \ldots, 0)$ has a 1 in the i^{th} place.

We will work with a scalar-valued function $f : \mathbb{R}^n \to \mathbb{R}$, so that $f(\vec{x}) = f(x_1, \ldots, x_n)$ is a number.

2 Definition of Limit. $\lim_{\vec{x}\to\vec{c}} f(\vec{x}) = L$ means that any output error tolerance of $f(\vec{x})$ away from L is guaranteed by some input error tolerance of $\vec{x}$ from $\vec{c}$. That is, for any output tolerance $\epsilon > 0$, there is some input tolerance $\delta > 0$ with:

$$0 < |\vec{x} - \vec{c}| < \delta \quad \Longrightarrow \quad |f(\vec{x}) - L| < \epsilon,$$

where "$\Longrightarrow$" means "implies" or "guarantees".

Note that the definition makes sense even if $f(\vec{x})$ is only defined for $\vec{x}$ in some punctured open ball around $\vec{c}$, for $0 < |\vec{x} - \vec{c}| < r$.

3 Limit Laws. If $\lim_{\vec{x}\to\vec{c}} f(\vec{x}) = L$ and $\lim_{\vec{x}\to\vec{c}} g(\vec{x}) = M$, then:

(a) $\displaystyle\lim_{\vec{x}\to\vec{c}} f(x) + g(x) \;=\; L + M$

(b) $\displaystyle\lim_{\vec{x}\to\vec{c}} f(x)g(x) \;=\; LM$

(c) $\displaystyle\lim_{\vec{x}\to\vec{c}} f(x)/g(x) \;=\; L/M,$ provided $M \neq 0$

Proof of (b). By hypothesis, we assume that any desired output errors for $f(\vec{x})$ and $g(\vec{x})$ can be guaranteed by some input error for $\vec{x}$. We proceed to find an upper bound (ceiling) for the output error of $f(\vec{x})g(\vec{x})$ in terms of the controllable errors of $f(\vec{x})$ and $g(\vec{x})$.

$$\begin{aligned} |f(\vec{x})g(\vec{x}) - LM| &= |f(\vec{x})g(\vec{x}) - f(\vec{x})M + f(\vec{x})M - LM| \\ &\leq |f(\vec{x})g(\vec{x}) - f(\vec{x})M| + |f(\vec{x})M - LM| \quad \text{because } |u+v| \leq |u| + |v| \\ &= |f(\vec{x})|\,|g(\vec{x}){-}M| + |M|\,|f(\vec{x}){-}L|. \end{aligned}$$

Now, given a desired $\epsilon > 0$, choose δ small enough that $0 < |\vec{x} - \vec{c}| < \delta$ guarantees:

$$|f(\vec{x}) - L| < 1, \qquad |f(\vec{x}) - L| < \frac{\epsilon}{2(|L|+1)}, \qquad |g(\vec{x}) - M| < \frac{\epsilon}{2(|M|+1)}.$$

The first inequality assures $|f(x)| < |L| + 1$, so our previous estimate guarantees:

$$|f(\vec{x})g(\vec{x}) - LM| \;\leq\; (|L|{+}1)\frac{\epsilon}{2(|L|+1)} \;+\; (|M|{+}1)\frac{\epsilon}{2(|M|+1)} \;=\; \epsilon.$$

We conclude that any desired output tolerance $\epsilon > 0$ for $f(\vec{x})g(\vec{x})$ can be guaranteed by some input tolerance δ, which is the definition of $\lim_{\vec{x}\to\vec{c}} f(\vec{x})g(\vec{x}) = LM$. $\square$

4 Continuity. We say that $f(\vec{x})$ is continuous at $\vec{x} = \vec{c}$ when $\lim_{\vec{x}\to\vec{c}} f(\vec{x}) = f(\vec{c})$.

Proposition: For functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, if $f(\vec{x})$ is continuous at $\vec{x} = \vec{c}$, and $g(y)$ is continous at $y = f(\vec{c})$, then the composition $g(f(\vec{x}))$ is continuous at $\vec{x} = \vec{c}$.

Proof. By hypothesis, we assume that the output errors of $f(\vec{x})$ and $g(y)$ can be controlled near $\vec{x} = \vec{c}$ and $y = f(\vec{c})$. We chain these error bounds together to control the error of $g(f(\vec{x}))$.

Given $\epsilon > 0$, choose $\delta' > 0$ so that $|y - f(\vec{c})| < \delta'$ guarantees $|g(y) - g(f(\vec{c}))| < \epsilon$. Now, taking δ' as an output error tolerance for $f(\vec{x})$, choose $\delta > 0$ so that $|\vec{x} - \vec{c}| < \delta$ guarantees $|f(\vec{x}) - f(\vec{c})| < \delta'$. Then we have:

$$|\vec{x} - \vec{c}| < \delta \quad \Longrightarrow \quad |f(\vec{x}) - f(\vec{c})| < \delta' \quad \Longrightarrow \quad |g(f(\vec{x})) - g(f(\vec{c}))| < \epsilon,$$

which means $\lim_{\vec{x}\to\vec{c}} g(f(\vec{x})) = g(f(\vec{c}))$. Conclusion: $g(f(\vec{x}))$ is continuous at $\vec{c}$. □

5 Linear Mappings. We say a function $\ell : \mathbb{R}^n \to \mathbb{R}$ is a *linear mapping* if $\ell(s\vec{u} + t\vec{v}) = s\,\ell(\vec{u}) + t\,\ell(\vec{v})$ for all vectors $\vec{u}, \vec{v}$ and scalars s, t. Thus we have:

$$\ell(\vec{x}) \;=\; \ell(x_1\vec{e}_1 + \cdots x_n\vec{e}_n) \;=\; x_1\ell(\vec{e}_1) + \cdots + x_n\ell(\vec{e}_n) \;=\; \vec{\ell}\cdot\vec{x},$$

the dot product of $\vec{x}$ with the vector $\vec{\ell} = (\ell(\vec{e}_1), \ldots, \ell(e_n))$. In linear algebra, we call $\vec{\ell}$ the *matrix* of the linear mapping $\ell(\vec{x})$.

An *affine mapping* means a linear mapping plus a constant: $m(\vec{x}) = \ell(\vec{x}) + b$.

6 Definition of Derivative. The *derivative* of a function $f : \mathbb{R}^n \to \mathbb{R}$ at $\vec{x} = \vec{c}$ is a linear mapping $Df_{\vec{c}} : \mathbb{R}^n \to \mathbb{R}$ which gives a very accurate affine approximation:

$$f(\vec{x}) \;\approx\; f(\vec{c}) + Df_{\vec{c}}(\vec{x} - \vec{c}) \qquad \text{for} \qquad \vec{x} \approx \vec{c}.$$

The error in the approximation must be vanishingly small relative to $|\vec{x} - \vec{c}|$:

$$\lim_{\vec{x}\to\vec{c}} \frac{|f(\vec{x}) - f(\vec{c}) - Df_{\vec{c}}(\vec{x} - \vec{c})|}{|\vec{x} - \vec{c}|} \;=\; 0.$$

7 Partial Derivative Theorem: If the functions $\frac{\partial f}{\partial x_1}(\vec{x}), \ldots, \frac{\partial f}{\partial x_n}(\vec{x})$ exist and are continuous near $\vec{x} = \vec{c}$, then $Df_{\vec{c}}(\vec{x})$ exists and equals the dot product of $\vec{x}$ with the gradient vector:

$$Df_{\vec{c}}(\vec{x}) = \nabla f(\vec{c}) \cdot \vec{x} \quad \text{where} \quad \nabla f(\vec{c}) = \left(\tfrac{\partial f}{\partial x_1}(\vec{c}), \ldots, \tfrac{\partial f}{\partial x_n}(\vec{c})\right).$$

Proof: For notational simplicity, we prove only the case $n = 2$, but the general case is completely analogous. We let $\vec{x} = (x, y)$ and $\vec{c} = (a, b)$. As part of the hypothesis, we assume the following limit values:

$$\lim_{x\to a} \tfrac{f(x,b) - f(a,b)}{x - a} = \tfrac{\partial f}{\partial x}(a, b), \qquad \lim_{y\to b} \tfrac{f(x,y) - f(x,b)}{y - b} = \tfrac{\partial f}{\partial y}(x, b), \qquad \lim_{x\to a} \tfrac{\partial f}{\partial y}(x, b) = \tfrac{\partial f}{\partial y}(a, b).$$

To prove the conclusion, we must show the accuracy of the approximation:

$$f(x,y) \quad\approx\quad f(a,b) + \tfrac{\partial f}{\partial x}(a,b)\,(x{-}a) + \tfrac{\partial f}{\partial y}(a,b)\,(y{-}b).$$

Specifically, we will find upper bounds for the error in terms of the controllable errors coming from the above three limits. We have:

$$\frac{\left|f(x,y) - f(a,b) - \tfrac{\partial f}{\partial x}(a,b)\,(x{-}a) - \tfrac{\partial f}{\partial y}(a,b)\,(y{-}b)\right|}{|(x,y)-(a,b)|}$$

$$= \quad \frac{\left|f(x,y) - f(x,b) - \tfrac{\partial f}{\partial y}(a,b)\,(y{-}b) + f(x,b) - f(a,b) - \tfrac{\partial f}{\partial x}(a,b)\,(x{-}a)\right|}{|(x{-}a,\ y{-}b)|}$$

$$\leq \quad \frac{\left|f(x,y) - f(x,b) - \tfrac{\partial f}{\partial y}(a,b)\,(y{-}b)\right|}{|(x{-}a,\ y{-}b)|} + \frac{\left|f(x,b) - f(a,b) - \tfrac{\partial f}{\partial x}(a,b)\,(x{-}a)\right|}{|(x{-}a,\ y{-}b)|}$$

$$\leq \quad \frac{\left|f(x,y) - f(x,b) - \tfrac{\partial f}{\partial y}(a,b)\,(y{-}b)\right|}{|y-b|} + \frac{\left|f(x,b) - f(a,b) - \tfrac{\partial f}{\partial x}(a,b)\,(x{-}a)\right|}{|x-a|}$$

$$\leq \quad \left|\frac{f(x,y)-f(x,b)}{y-b} - \frac{\partial f}{\partial y}(a,b)\right| + \left|\frac{f(x,b)-f(a,b)}{x-a} - \frac{\partial f}{\partial x}(a,b)\right|$$

$$\leq \quad \left|\frac{f(x,y)-f(x,b)}{y-b} - \frac{\partial f}{\partial y}(x,b)\right| + \left|\frac{\partial f}{\partial y}(x,b) - \frac{\partial f}{\partial y}(a,b)\right| + \left|\frac{f(x,b)-f(a,b)}{x-a} - \frac{\partial f}{\partial x}(a,b)\right|$$

The fourth line follows because $|(x{-}a, y{-}b)| \geq |x{-}a|, |y{-}b|$; and in the last line, we used the triangle inequality $|u-v| \leq |u-w| + |w-v|$ for $w = \tfrac{\partial f}{\partial y}(x,b)$.

Now the three terms at the end of the above estimate all go to zero as $(x,y) \to (a,b)$, by the three limit values mentioned above. (Specifically, there is some input error $|(x,y)-(a,b)| < \delta$ which guarantees that each of the three terms is less than $\frac{\epsilon}{3}$, so that the total output error is less than ϵ.) Thus we conclude:

$$\lim_{(x,y)\to(a,,b)} \frac{\left|f(x,y) - f(a,b) - \tfrac{\partial f}{\partial x}(a,b)\,(x{-}a) - \tfrac{\partial f}{\partial y}(a,b)\,(y{-}b)\right|}{|(x,y)-(a,b)|} \;=\; 0,$$

which means by definition that $Df_{(a,b)}(x,y) = \nabla f(a,b)\cdot(x,y)$. □

Note that the proof only needed the continuity of one of the partial derivatives. For general n, we need $n{-}1$ of them continuous at $\vec{x} = \vec{c}$.

8 Derivative Product Theorem: If the functions $f, g : \mathbb{R}^n \to \mathbb{R}$ are differentiable at $\vec{x} = \vec{c}$, then the product function $f(\vec{x})\, g(\vec{x})$ has derivative mapping:

$$D(fg)_{\vec{c}}(\vec{x}) \;=\; f(\vec{c})\, Df_{\vec{c}}(\vec{x}) \;+\; g(\vec{c})\, Dg_{\vec{c}}(\vec{x}).$$

Proof. We must show the accuracy of the approximation:

$$f(\vec{x})\, g(\vec{x}) \;\approx\; f(\vec{c})\, g(\vec{c}) \;+\; f(\vec{c})\, Dg_{\vec{c}}(\vec{x}-\vec{c}) \;+\; g(\vec{c})\, Df_{\vec{c}}(\vec{x}-\vec{c}).$$

We have:

$$\frac{|f(\vec{x})g(\vec{x}) \;-\; f(\vec{c})g(\vec{c}) \;-\; f(\vec{c})\, Dg_{\vec{c}}(\vec{x}-\vec{c}) \;-\; g(\vec{c})\, Df_{\vec{c}}(\vec{x}-\vec{c})|}{|\vec{x}-\vec{c}|}$$

$$= \;\frac{|f(\vec{x})g(\vec{x}) \;-\; f(\vec{x})g(\vec{c}) \;-\; f(\vec{c})\, Dg_{\vec{c}}(\vec{x}-\vec{c}) \;+\; f(\vec{x})g(\vec{c}) \;-\; f(\vec{c})g(\vec{c}) \;-\; g(\vec{c})\, Df_{\vec{c}}(\vec{x}-\vec{c})|}{|\vec{x}-\vec{c}|}$$

$$\le \;\frac{|f(\vec{x})g(\vec{x}) \;-\; f(\vec{x})g(\vec{c}) \;-\; f(\vec{c})\, Dg_{\vec{c}}(\vec{x}-\vec{c})|}{|\vec{x}-\vec{c}|} \;+\; \frac{|f(\vec{x})\, g(\vec{c}) \;-\; f(\vec{c})\, g(\vec{c}) \;-\; g(\vec{c})\, Df_{\vec{c}}(\vec{x}-\vec{c})|}{|\vec{x}-\vec{c}|}$$

$$\le \;\frac{|f(\vec{x})g(\vec{x}) \;-\; f(\vec{x})g(\vec{c}) \;-\; f(\vec{x})\, Dg_{\vec{c}}(\vec{x}-\vec{c})|}{|\vec{x}-\vec{c}|} \;+\; \frac{|f(\vec{x})\, Dg_{\vec{c}}(\vec{x}-\vec{c}) \;-\; f(\vec{c})\, Dg_{\vec{c}}(\vec{x}-\vec{c})|}{|\vec{x}-\vec{c}|}$$
$$+\; \frac{|f(\vec{x}) \;-\; f(\vec{c}) \;-\; Df_{\vec{c}}(\vec{x}-\vec{c})|}{|\vec{x}-\vec{c}|}\, |g(\vec{c})|$$

$$= \;|f(\vec{x})|\, \frac{|g(\vec{x}) \;-\; g(\vec{c}) \;-\; Dg_{\vec{c}}(\vec{x}-\vec{c})|}{|\vec{x}-\vec{c}|} \;+\; |f(\vec{x}) - f(\vec{c})|\, \frac{|Dg_{\vec{c}}(\vec{x}-\vec{c})|}{|\vec{x}-\vec{c}|}$$
$$+\; \frac{|f(\vec{x}) \;-\; f(\vec{c}) \;-\; Df_{\vec{c}}(\vec{x}-\vec{c})|}{|\vec{x}-\vec{c}|}\, |g(\vec{c})|$$

Now each of the three terms on the last line goes to zero, because by hypothesis $f(\vec{x}) \approx f(c) + Df_{\vec{c}}(\vec{x}-\vec{c})$, $\;g(\vec{x}) \approx g(\vec{c}) + Dg_{\vec{c}}(\vec{x}-\vec{c})$, and:

$$\frac{|Df_{\vec{c}}(\vec{x}-\vec{c})|}{|\vec{x}-\vec{c}|} \;=\; \left|\nabla f(\vec{c}) \cdot \frac{\vec{x}-\vec{c}}{|\vec{x}-\vec{c}|}\right| \;\le\; |\nabla f(\vec{c})|\,.$$