Math 254H **Centroid Theorem** **Jan 11, 2016**

A *median* of a triangle is the line segment from a vertex to the midpoint of the opposite side.

PROPOSITION: Given any triangle in the plane, the three medians intersect at a common point, which is $\frac{2}{3}$ of the way along each median.

PROOF: We translate the proposition into the language of vector algebra. Let $\vec{u}, \vec{v}, \vec{w}$ be the vectors from the origin to the vertices of the triangle..

The vector from $\vec{v}$ to $\vec{w}$ is $\vec{w} - \vec{v}$, and the midpoint of the corresponding side is:

$$\vec{v} + \tfrac{1}{2}(\vec{w} - \vec{v}) \;=\; \tfrac{1}{2}\vec{v} + \tfrac{1}{2}\vec{w}.$$

The vector from $\vec{u}$ to this midpoint is $\frac{1}{2}\vec{v} + \frac{1}{2}\vec{w} - \vec{u}$, and the median from $\vec{u}$ to the midpoint is the parametrized line segment:

$$\vec{\ell}(t) = \vec{u} + t(\tfrac{1}{2}\vec{v} + \tfrac{1}{2}\vec{w} - \vec{u}) \quad \text{for} \quad 0 \leq t \leq 1.$$

(That is, the points of the segment are the endpoints of the vectors $\vec{\ell}(t)$, written in standard form from the origin.) The point $\frac{2}{3}$ of the way along this median is:

$$\begin{aligned}
\vec{\ell}(\tfrac{2}{3}) &= \vec{u} + \tfrac{2}{3}(\tfrac{1}{2}\vec{v} + \tfrac{1}{2}\vec{w} - \vec{u}) \\
&= \vec{u} + \tfrac{1}{3}\vec{v} + \tfrac{1}{3}\vec{w} - \tfrac{2}{3}\vec{u} \\
&= (1 - \tfrac{2}{3})\vec{u} + \tfrac{1}{3}\vec{v} + \tfrac{1}{3}\vec{w} \\
&= \tfrac{1}{3}(\vec{u} + \vec{v} + \vec{w}),
\end{aligned}$$

where we expand and factor using the distributive property of scalar multiplication over vector addition.

The same computation for the other two medians gives their $\frac{2}{3}$ points:

$$\begin{aligned}
\vec{v} + \tfrac{2}{3}(\tfrac{1}{2}\vec{u} + \tfrac{1}{2}\vec{w} - \vec{v}) &= \tfrac{1}{3}(\vec{u} + \vec{v} + \vec{w}), \\
\vec{w} + \tfrac{2}{3}(\tfrac{1}{2}\vec{u} + \tfrac{1}{2}\vec{v} - \vec{w}) &= \tfrac{1}{3}(\vec{u} + \vec{v} + \vec{w}).
\end{aligned}$$

Thus, all three medians contain a common point, the $\frac{2}{3}$ point for each. $\square$

NOTE: We call this common point the *centroid* of the triangle. The proof shows it is the vector average of $\vec{u}, \vec{v}, \vec{w}$.