Improper Integrals

Integrals near a vertical asymptote. What happens if we take the integral of a function over an interval containing a vertical asymptote, such as:

$$I \;=\; \int_0^2 \frac{1}{x}\,dx \;=\; ??$$

Algebraically, we would get $I = \ln|2| - \ln|0|$, but $\ln(0)$ is undefined. Numerically, the Riemann sum for I does not converge, because of the very large values of $f(x)$ near $x = 0$. Geometrically, I measures a region (the positive area in the graph on the next page) which stretches infinitely along the asymptote $x = 0$, and the meaning of such an infinitely extended area is not clear.

Our previous definitions fail to give meaning to this integral, so we give a new definition:

$$\int_0^2 \frac{1}{x}\,dx \;=\; \lim_{r\to 0^+} \int_r^2 \frac{1}{x}\,dx.$$

That is, we take the integral over the interval $x \in [r, 2]$ where the function is continuous, then take the limit as r squeezes up against the asymptote $x = 0$ from the right. Now, $\int_r^2 \frac{1}{x}\,dx = \ln|2| - \ln|r|$, and $\lim_{r\to 0^+} \ln(r) = -\infty$, meaning $\ln(r)$ becomes a larger and larger negative number, so the improper integral is:*

$$\int_0^2 \frac{1}{x}\,dx \;=\; \lim_{r\to 0^+} \ln(2) - \ln(r) \;=\; \infty.$$

This says that the total area under the graph $y = \frac{1}{x}$ and above $[0, 2]$ is infinite: no matter how many square units of paint are put on this region, there will still be unpainted area high up next to the asymptote.

General definition: If the function $f(x)$ has a vertical asymptote near $x = q$, we define the *improper integral of vertical type*:

- on an interval $[a, q]$ as $\displaystyle\int_a^q f(x)\,dx \;=\; \lim_{r\to q^-} \int_a^r f(x)\,dx;$
- on an interval $[q, b]$ as $\displaystyle\int_q^b f(x)\,dx \;=\; \lim_{r\to q^+} \int_r^b f(x)\,dx.$
- on an interval with $q \in (a, b)$ as $\displaystyle\int_a^b f(x)\,dx \;=\; \int_a^q f(x)\,dx + \int_q^b f(x)\,dx.$

If such an integral has a finite value, we say it *converges*; if it is infinite or undefined, we say it *diverges*.

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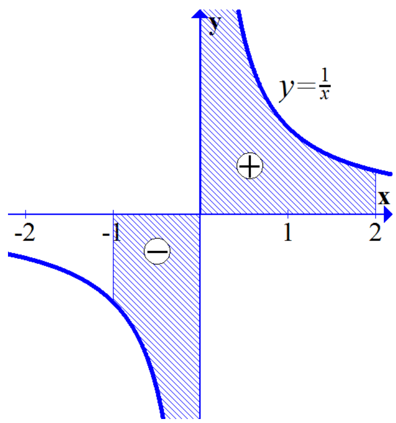
*We take $\ln(2)$ minus a larger and larger negative number; and this equals a larger and larger positive number, denoted by ∞.

EXAMPLE: Evaluate $\int_{-1}^{2}\frac{1}{x}\,dx$. This attempts to measure two infinite regions: one above $[0,2]$ along the positive y-axis, and another below $[-1,0]$ along the negative y-axis.



The improper integral avoids the asymptote from both sides:

$$\int_{-1}^{2}\frac{1}{x}\,dx \;=\; \int_{-1}^{0}\frac{1}{x}\,dx+\int_{0}^{2}\frac{1}{x}\,dx \;=\; \lim_{r\to 0^-}\int_{-1}^{r}\frac{1}{x}\,dx \;+\; \lim_{r\to 0^+}\int_{r}^{2}\frac{1}{x}\,dx.$$

But when we try to calculate this, we get:

$$\int_{-1}^{2}\frac{1}{x}\,dx \;=\; \left(\lim_{r\to 0^-}\ln|r|-\ln|-1|\right) \;+\; \left(\lim_{r\to 0^+}\ln|2|-\ln|r|\right) \;=\; -\infty+\infty\,,$$

which is an indeterminate form: the integral is truly undefined. We have no clear meaning for an infinite positive area canceled by an infinite negative area. In particular, the naive answer is wrong:

$$\int_{-1}^{2}\frac{1}{x}\,dx \;=\; \text{undefined} \;\neq\; \ln|2|-\ln|-1|.$$

EXAMPLE: Evaluate $\int_{1}^{2}\frac{1}{\sqrt{x-1}}\,dx$. Since the vertical asymptote is $x=1$, we have:

$$\int_{1}^{2}\frac{1}{\sqrt{x-1}}\,dx \;=\; \lim_{r\to 1^+}\int_{r}^{2}\frac{1}{\sqrt{x-1}}\,dx \;=\; \lim_{r\to 1^+} 2\sqrt{x-1}\,\Big|_{x=r}^{x=2}$$

$$=\; \lim_{r\to 1^+} 2\sqrt{2-1}-2\sqrt{r-1} \;=\; 2-0 \;=\; 2.$$

In this case, the region has a finite area of 2, even though it stretches infinitely high along the vertical asymptote. Thus, if we have enough paint for 2 square units, and paint higher and higher parts of the region using less and less paint, we never run out.

Integrals near a horizontal asymptote. If $y = f(x)$ has $y = 0$ as a horizontal asymptote, we can define *improper integrals of horizontal type.*

- If $\lim_{x\to\infty} f(x) = 0$, we define the integral on an interval $[a, \infty)$ as:

$$\int_a^\infty f(x)\,dx \;=\; \lim_{r\to\infty} \int_a^r f(x)\,dx.$$

- If $\lim_{x\to\infty} f(x) = 0$, we define the integral over an interval $(-\infty, a]$ as:

$$\int_{-\infty}^a f(x)\,dx \;=\; \lim_{r\to-\infty} \int_r^a f(x)\,dx.$$

- If $\lim_{x\to\pm\infty} f(x) = 0$, we define its integral over the whole real line $(-\infty, \infty)$ by splitting at any finite value $x = a$:

$$\int_{-\infty}^\infty f(x)\,dx \;=\; \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx \qquad \text{for any } a.$$

EXAMPLE: $\int_1^\infty \frac{1}{x^2}\,dx \;=\; \lim\limits_{r\to\infty} \int_1^r \frac{1}{x^2}\,dx \;=\; \lim\limits_{r\to\infty} \left(-\frac{1}{x}\right)\Big|_{x=1}^{x=r} \;=\; \lim_{r\to\infty} -\frac{1}{r} + \frac{1}{1} \;=\; 1\,.$
This integral measures a region which stretches infinitely along the x-axis above $[1, \infty)$, but which has a finite total area of 1.

On the other hand $\int_1^\infty \frac{1}{\sqrt{x}}\,dx \;=\; \lim_{r\to\infty} 2\sqrt{x}\,\big|_{x=1}^{x=r} \;=\; \infty$. In fact, $\int_1^\infty \frac{1}{x^p}\,dx$ is finite if $p > 1$, but is infinite if $p \leq 1$. Informally, the faster $f(x)$ shrinks as $x \to \infty$, the easier it is for the integral to converge to a finite value.

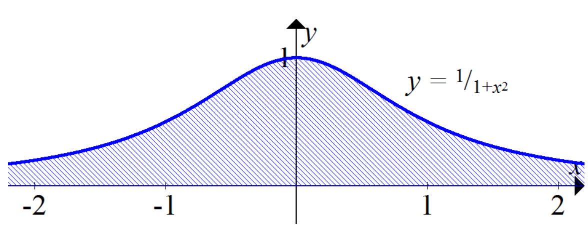
EXAMPLE: $\int_0^\infty e^{-x}\,dx = \lim_{r\to\infty} \int_0^r e^{-x}\,dx = \lim_{r\to\infty} -e^{-x}\big|_{x=0}^{x=r} = -0 - (-1) = 1.$
It is not surprising that this converges, because e^{-x} shrinks faster than $\frac{1}{x^p}$ for any p.

EXAMPLE:

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx \;=\; \lim_{r\to-\infty} \int_r^0 \frac{1}{1+x^2}\,dx \;+\; \lim_{r\to\infty} \int_0^r \frac{1}{1+x^2}\,dx$$

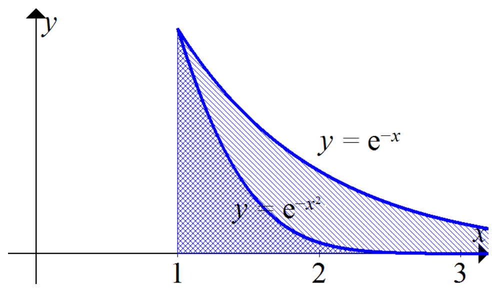
$$= \lim_{r\to-\infty} \tan^{-1}(x)\Big|_{x=r}^{x=0} \;+\; \lim_{r\to\infty} \tan^{-1}(x)\Big|_{x=0}^{x=r} \;=\; \left(0 - (-\tfrac{\pi}{2})\right) + \left(\tfrac{\pi}{2} - 0\right) \;=\; \pi.$$

Remarkably, the total area under $y = \frac{1}{1+x^2}$ turns out to be π, same as a unit circle!



Comparison tests for convergence. Sometimes an improper integral is too complicated to find an algebraic antiderivative, but we can still be sure it converges because the infinite region measured fits inside a larger region of known finite area.

For example, the Gaussian bell-curve integral $\int_1^\infty e^{-x^2}\,dx$ cannot be integrated by an antiderivative. However, for $x \geq 1$, we have $x^2 \geq x$, so $e^{-x^2} \leq e^{-x}$: that is, the curve $y = e^{-x^2}$ lies below $y = e^{-x}$:



We can easily evaluate the area below the upper curve, which shows that the smaller area under the lower curve is finite, i.e. the improper integral converges:

$$\int_1^\infty e^{-x^2}\,dx \;<\; \int_1^\infty e^{-x}\,dx \;=\; 0-(-e^{-1}) \;=\; \tfrac{1}{e} \;\approx\; 0.37\;.$$

Direct Comparison Test: Consider an improper integral $\int_a^b g(x)$, with a or b infinite.

- If $|f(x)| \leq g(x)$ for $x \in [a,b]$, and $\int_a^b g(x)\,dx$ converges, then $\int_a^b f(x)\,dx$ converges.
- If $f(x) \geq g(x) \geq 0$ for $x \in [a,b]$ and $\int_a^b g(x)\,dx$ diverges, then $\int_a^b f(x)\,dx$ diverges.

The proof uses the Domination Rule for ordinary integrals (§4.2), plus some complications with limits.

EXAMPLE: Does $\int_0^\infty \frac{4\sin(x)+1}{e^{2x}+x^2}\,dx$ converge? This function shrinks rapidly, since the top does not grow, and the bottom grows exponentially; thus we guess that the integral converges. To prove this using the first part of the Test, we should bound $f(x) = \frac{4\sin(x)+1}{e^{2x}+x^2}$ inside the graph of a fairly simple comparison function $g(x) = \frac{g_1(x)}{g_2(x)}$ with $|f(x)| \leq g(x)$. Now, increasing the numerator of $f(x)$ and decreasing its denominator gives a larger fraction, so we take:

$$\left|\frac{4\sin(x)+1}{e^{2x}+x^2}\right| \;\leq\; \frac{5}{e^{2x}} \;=\; 5e^{-2x}\,.$$

The comparison integral converges: $\int_0^\infty 5e^{-2x}\,dx = \lim_{r\to\infty}\left(-\frac{5}{2}e^{-2x}\right)\Big|_{x=0}^{x=r} = \frac{5}{2}$; hence the original integral also converges:

$$\left|\int_0^\infty \frac{4\sin(x)+1}{e^{2x}+x^2}\,dx\right| \;\leq\; \tfrac{5}{2}\,.$$

By contrast, to prove *divergence* of a fractional $f(x)$, we would bound $f(x) \geq g(x)$, above a floor function $g(x)$ with smaller numerator and larger denominator, and $\int g(x)\,dx = \infty$.

Limit Comparison Test or *Ratio Comparison Test:* For functions $f(x), g(x)$ with $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = L \neq 0$, the improper integral $\int_a^\infty f(x)\,dx$ converges if and only if $\int_a^\infty g(x)\,dx$ converges.

In the case that $g(x) \geq 0$, this is simply because, given $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$, we can take x large enough so that $\frac{1}{2}Lg(x) \leq f(x) \leq \frac{3}{2}Lg(x)$, and we can apply the Direct Comparison Test.

To apply this Test to $\int_a^\infty f(x)\,dx$ for a fraction $f(x) = \frac{f_1(x)}{f_2(x)}$, we generally choose the comparison function $g(x) = \frac{g_1(x)}{g_2(x)}$ where $g_1(x)$ is the largest term in $f_1(x)$, and likewise with $g_2(x)$ and $f_2(x)$. For example, for:

$$f(x) = \frac{x^2 - e^{-x} + \sin(x)}{\sqrt{x^5 + 7}} \qquad \text{take} \qquad g(x) = \frac{x^2}{\sqrt{x^5}} = x^{-1/2}\,,$$

and we easily see $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ (see §6.8). We previously showed $\int_a^\infty x^{-1/2}\,dx$ diverges, so the original integral $\int_a^\infty f(x)\,dx$ also diverges.