Partial Fractions

Integrating basic rational functions. For a function $f(x)$, we have examined several algebraic methods* for finding its indefinite integral (antiderivative) $F(x) = \int f(x)\,dx$, which allows us to compute definite integrals $\int_a^b f(x)\,dx = F(b) - F(a)$ by the Second Fundamental Theorem.

In this section, we will learn a special technique to integrate any *rational function*, meaning a quotient of two polynomials:

$$f(x) \;=\; \frac{g(x)}{h(x)} \;=\; \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0}\,,$$

where a_i, b_j are constant coefficients. We call the largest powers m and n the *degrees* of the polynomials $g(x)$ and $h(x)$, assuming that the highest coefficients $a_m, b_n \neq 0$.

We have several basic rational functions whose integrals we already know:

(i) $\displaystyle\int a_m x^m + \cdots + a_1 x + a_0 \; dx \;=\; \tfrac{a_m}{m+1}x^{m+1} + \cdots + \tfrac{a_1}{2}x^2 + a_0 x + C.$

(ii) $\displaystyle\int \frac{1}{x-a}\,dx \;=\; \ln|x-a| + C.$

(iii) $\displaystyle\int \frac{1}{(x-a)^n}\,dx \;=\; -\frac{1}{(n-1)(x-a)^{n-1}}$ for $n \geq 2$.

(iv) $\displaystyle\int \frac{x}{x^2+a}\,dx \;=\; \tfrac{1}{2}\int \frac{1}{x^2+a}\cdot 2x\,dx \;=\; \tfrac{1}{2}\ln|x^2+a| + C$

(v) $\displaystyle\int \frac{1}{x^2+a}\,dx \;=\; \tfrac{1}{\sqrt{a}}\int \frac{1}{(\frac{x}{\sqrt{a}})^2+1}\cdot \tfrac{1}{\sqrt{a}}\,dx \;=\; \tfrac{1}{\sqrt{a}}\arctan(\tfrac{x}{\sqrt{a}}) + C,$ for $a > 0$.

(vi) $\displaystyle\int \frac{1}{(x^2+1)^2}\,dx.$ Letting $x = \tan(\theta)$, $x^2+1 = \sec^2(\theta)$, $dx = \sec^2(\theta)\,d\theta$:

$$\int \frac{1}{(x^2+1)^2}\,dx \;=\; \int \frac{1}{\sec^4(\theta)}\,\sec^2(\theta)\,d\theta \;=\; \int \cos^2(\theta)\,d\theta$$

$$=\; \tfrac{1}{2}(\theta + \sin(\theta)\cos(\theta)) + C \;=\; \tfrac{1}{2}\left(\arctan(x) + \frac{x}{x^2+1}\right) + C\,.$$

We used: $\int \cos^2(\theta)\,d\theta = \int \frac{1}{2} + \frac{1}{2}\cos(2\theta)\,d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) = \frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cos(\theta)$, and (as in §6.6) $\sin(\theta) = \frac{x}{\sqrt{x^2+1}}$, $\cos(\theta) = \frac{1}{\sqrt{x^2+1}}$.

Quadratic denominator. With the above basic integrals, we can integrate any rational function with numerator of degree at most 1 and denominator of degree at most 2:

$$\int \frac{px+q}{ax^2+bx+c}\,dx\,.$$

There are two different cases, depending on the sign of the *discriminant* $d = b^2 - 4ac$.

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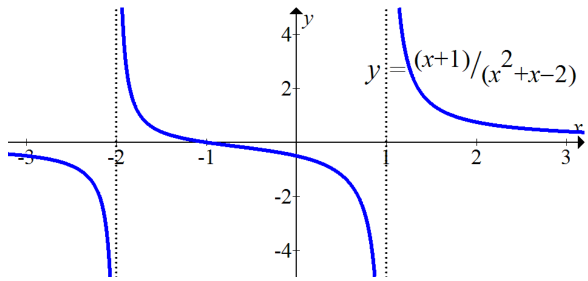
*Substitution §4.5, Integration by Parts §7.1, Products of Trig Functions §7.2, Reverse Trig Substitution §7.3

EXAMPLE: Here is how to handle the case where $d = b^2 - 4ac > 0$, such as:

$$\int \frac{x+1}{x^2+x-2}\, dx,$$

where $d = 1^2 - 4(1)(-2) = 9$. By the Quadratic Formula, the denominator has two real roots $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = 1, -2$, which are the vertical asymptotes of our function:



We split our function into a sum of simple parts, each having just one vertical asymptote:

$$\frac{x+1}{x^2+x-2} \;=\; \frac{x+1}{(x-1)(x+2)} \;=\; \frac{A}{x-1} + \frac{B}{x+2}\,.$$

This is called the *partial fraction expansion* of our rational function. For any constants A, B, the graph of the right-hand function will have the same asymptotes as our original function, but we can actually find constants which make the two exactly equal. Clearing denominators, we want A, B such that:

$$x+1 \;=\; A(x+2) + B(x-1) \quad \text{for all } x.$$

Setting $x = 1$ gives $1+1 = A(1+2) + B(0)$, so $A = \frac{2}{3}$; and setting $x = -2$ gives $-2+1 = A(0) + B(-2-1)$, so $B = \frac{1}{3}$. Now we can use the basic integral (ii) above:

$$\int \frac{x+1}{x^2+x-2}\, dx \;=\; \int \frac{\frac{2}{3}}{x-1} + \frac{\frac{1}{3}}{x+2}\, dx \;=\; \tfrac{2}{3}\ln|x-1| + \tfrac{1}{3}\ln|x+2| + C\,.$$

EXAMPLE: The other case is when $d = b^2 - 4ac < 0$, such as:

$$\int \frac{x+1}{x^2+x+1}\, dx,$$

for which $d = 1^2 - 4(1)(1) = -3$. In this case, the denominator has no real-number zeroes: $x^2 + x + 1 > 0$, and it cannot be factored; it is an *irreducible* polynomial. The graph of $\frac{x+1}{x^2+x+1}$ has no vertical asymptotes. Our strategy is to reduce its integral to the basic integrals (iv) and (v) above.

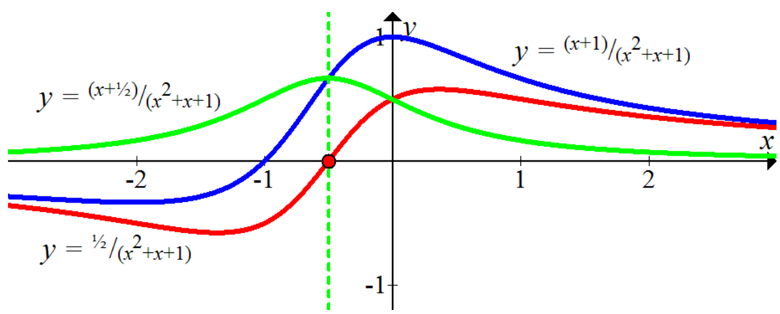
The first step is to complete the square in the denominator and force it into the form $(x+p)^2 + q$, the same process that produces the Quadratic Formula:

$$x^2+x+1 \;=\; x^2 + 2(\tfrac{1}{2})x + (\tfrac{1}{2})^2 - (\tfrac{1}{2})^2 + 1 \;=\; (x+\tfrac{1}{2})^2 + \tfrac{3}{4}\,.$$

Thus, letting $u = x + \frac{1}{2}$, $du = dx$:

$$\begin{aligned}
\int \frac{x+1}{x^2+x+1}\,dx &= \int \frac{(x+\frac{1}{2}) - \frac{1}{2} + 1}{(x+\frac{1}{2})^2 + \frac{3}{4}}\,dx \\
&= \int \frac{u}{u^2+\frac{3}{4}}\,dx + \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\,dx \\
&= \tfrac{1}{2}\ln|u^2+\tfrac{3}{4}| + \tfrac{1}{2\sqrt{3/4}}\arctan(\tfrac{u}{\sqrt{3/4}}) + C \\
&= \tfrac{1}{2}\ln\big|x^2+x+1\big| + \tfrac{1}{\sqrt{3}}\arctan(\tfrac{2x+1}{\sqrt{3}}) + C.
\end{aligned}$$

In fact, the two terms in our answer correspond to splitting the original function (blue) into a graph with reflection symmetry across the line $x = -\frac{1}{2}$ (green), and a graph with $180°$ rotation symmetry around the point $(-\frac{1}{2}, 0)$ (red):



EXAMPLE: One more case: if the numerator has degree greater than or equal to the denominator, for example:

$$\int \frac{x^4+2x+3}{x^2+x-2}\,dx\,.$$

Then $y = 0$ is no longer a horizontal asymptote. Instead, the behavior of the function as $x \to \pm\infty$ is controlled by a polynomial curve obtained by polynomial long division.

$$\begin{array}{r l}
 & x^2 - x + 3 \quad \text{rem } -3x+9 \\
x^2+x-2 \,\big)\!\! & x^4 \qquad\qquad\quad + 2x + 3 \\
 & -(x^4 + x^3 - 2x^2) \\
 & \qquad - x^3 + 2x^2 + 2x + 3 \\
 & \qquad -(x^3 - x + 2x) \\
 & \qquad\qquad 3x^2 \qquad + 3 \\
 & \qquad\quad -(3x^2 + 3x - 6) \\
 & \qquad\qquad\quad -3x + 9
\end{array}$$

Thus $x^4 + 2x + 3 = (x^2{-}x{+}3){\cdot}(x^2{+}x{-}2) + (-3x{+}9)$, and:

$$\frac{x^4+2x+3}{x^2+x-2} = (x^2{-}x{+}3) + \frac{-3x+9}{x^2+x+1} = (x^2{-}x{+}3) + \frac{2}{x-1} - \frac{5}{x+2}$$

The last equality is a partial fraction expansion similar to our first example above. Now:

$$\int \frac{x^4+2x+3}{x^2+x-2}\,dx = \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + 3x + 2\ln|x-1| - 5\ln|x+2| + C\,.$$

General case. The above techniques suffice to integrate any rational function $\int \frac{g(x)}{h(x)}\,dx$, provided we can factor the denominator $h(x)$. First, we perform a partial fraction decomposition of $f(x)$ into a sum of terms of the following forms:

- A polynomial $q(x)$, which is the quotient in the long division $g(x) \div h(x) = q(x)$ with remainder $r(x)$ of smaller degree than $h(x)$, so that $\frac{g(x)}{h(x)} = q(x) + \frac{r(x)}{h(x)}$.
- For each linear factor $x-c$ of the denominator $h(x)$, suppose $(x-a)^n$ is the highest power which divides $h(x)$. Then we add a sum of n terms:
$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}\,.$$
- For each irreducible quadratic factor ax^2+bx+c of $h(x)$, suppose $(ax^2+bx+c)^n$ is the highest power which divides $h(x)$. Then we add a sum of n terms:
$$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}\,.$$

Leaving aside the polynomial term $q(x)$, we set $r(x)/h(x)$ equal to the sum of all the other terms above, then we clear the denominators and solve for all the unknown constants in the numerators as we did for A, B in our first example above. (The *Partial Fractions Theorem* states that a unique solution always exists.) Once this is done, we can integrate using (i)–(vi) and the above examples (see also the integrals of higher powers below).

EXAMPLE: We find the partial fraction expansion of:

$$f(x) \;=\; \frac{1}{x^2(x^2+1)^2} \;=\; \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B_1x{+}C_1}{x^2+1} + \frac{B_2x{+}C_2}{(x^2+1)^2}\,.$$

Since the numerator has degree less than the denominator, there is no polynomial term $q(x)$. We need to find the six constants $A_1, A_2, B_1, B_2, C_1, C_2$ which make the above equation valid. Clearing denominators gives:

$$\begin{aligned} 1 \;&=\; A_1x(x^2{+}1)^2 + A_2(x^2{+}1)^2 + (B_1x{+}C_1)x^2(x^2{+}1) + (B_2x{+}C_2)x^2 \\ &=\; (A_1{+}B_1)x^5 + (A_2{+}C_1)x^4 + (2A_1{+}B_1{+}B_2)x^3 + (2A_2{+}C_1{+}C_2)x^2 + A_1x + A_2 \end{aligned}$$

Since this is an equality of polynomial functions, the coefficients of x^k on the right must equal the coefficients of $1 = 0x^5 + \cdots + 0x + 1$ on the left:

$$\begin{gathered} A_1 + B_1 = 0 \\ A_2 + C_1 = 0 \\ 2A_1 + B_1 + B_2 = 0 \\ 2A_2 + C_1 + C_2 = 0 \\ A_1 = 0, \quad A_2 = 1. \end{gathered}$$

We solve this as:

$$\begin{gathered} A_1 = 0, \quad A_2 = 1, \quad B_1 = -A_1 = 0, \quad C_1 = -A_2 = -1, \\ B_2 = -2A_1 - B_1 = 0, \quad C_2 = -2A_2 - C_1 = -1\,. \end{gathered}$$

Hence, according to (i)–(vi):

$$\begin{aligned} \int \frac{1}{x^2(x^2+1)^2}\,dx \;&=\; \int \frac{1}{x^2} - \frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}\,dx \\ &=\; -\frac{1}{x} - \arctan(x) - \frac{1}{2}\left(\arctan(x) + \frac{x}{x^2+1}\right). \end{aligned}$$

Trig integrals again. In §7.2–7.3, we reduced trig integrals by substitution to rational function integrals, which we can now find by partial fractions. For example:

$$\int \sec(x)\,dx \;=\; \int \frac{1}{\cos^2(x)}\cdot\cos(x)\,dx \;=\; \int \frac{1}{1-\sin^2(x)}\cdot\cos(x)\,dx$$

$$\left\{\begin{array}{c} u=\sin(x) \\ du=\cos(x)\,dx \end{array}\right\} \;=\; \int \frac{1}{1-u^2}\,du \;=\; \int \frac{\frac{1}{2}}{1+u}+\frac{\frac{1}{2}}{1-u}\,du$$

$$=\; \tfrac{1}{2}\ln(1{+}u)-\tfrac{1}{2}\ln(1{-}u) \;=\; \ln\sqrt{\frac{1{+}u}{1{-}u}} \;=\; \ln\sqrt{\frac{1{+}\sin(x)}{1{-}\sin(x)}}\,.$$

CHALLENGE: Show by identities that this is equal to our previous answer $\int\sec(x)\,dx = \ln|\tan(x)+\sec(x)|$ given in §7.2 . Also: try this method on $\int\sec^3(x)\,dx = \int\frac{1}{(1-u^2)^2}\,du$.

CHALLENGE: Integrate $\int\sqrt{\tan(x)}\,dx$. The substitution $u=\tan(x)$, $x=\arctan(u)$, $dx=\frac{1}{u^2+1}\,du$ gives $\int\frac{\sqrt{u}}{u^2+1}\,du$. Then $z=\sqrt{u}$ reduces this to a rational function. For partial fractions, factor $z^4+1=(z^2{+}1)^2-2z^2=(z^2+1+\sqrt{2}\,z)(z^2+1-\sqrt{2}\,z)$. Solve:

$$\frac{z^2}{z^4+1} \;=\; \frac{Az+B}{z^2+\sqrt{2}\,z+1}+\frac{Cz+D}{z^2-\sqrt{2}\,z+1}\,.$$

Higher quadratic powers. For powers of irreducible quadratics in the denominator:

$$(*)\qquad \int\frac{x}{(x^2+1)^n}\,dx \;=\; -\frac{1}{2(n{-}1)(1+x^2)^{n-1}},$$

using the substitution $u=x^2+1$. The other basic rational integral

$$I_n=\int\frac{1}{(x^2+1)^n}\,dx$$

can be evaluated *recursively* using Integration by Parts and the previous integral:

$$\begin{aligned}
I_n = \int\tfrac{1}{(x^2+1)^n}\,dx \;&=\; \int\tfrac{x^2+1}{(x^2+1)^n}\,dx-\int\tfrac{x^2}{(x^2+1)^n}\,dx && \text{since } 1=(x^2{+}1)-x^2\\
&= \int\tfrac{1}{(x^2+1)^{n-1}}\,dx-\int x\cdot\tfrac{x}{(x^2+1)^n}\,dx && \text{setting up Int By Parts}\\
&= \int\tfrac{1}{(x^2+1)^{n-1}}\,dx + x\cdot\tfrac{1}{2(n-1)(x^2+1)^{n-1}}-\int 1\cdot\tfrac{1}{2(n-1)(x^2+1)^{n-1}}\,dx && \text{by IBP and integral } (*)\\
&= I_{n-1}+\tfrac{x}{2(n-1)(1+x^2)^{n-1}}-\tfrac{1}{2(n-1)}I_{n-1} && \text{by definition of } I_{n-1}.
\end{aligned}$$

Simplifying, we get the recursive formula:

$$I_n \;=\; \frac{2n{-}3}{2(n{-}1)}I_{n-1}+\frac{x}{2(n{-}1)(x^2+1)^{n-1}}.$$

EXAMPLE: For $n=3$, we already know $I_2=\int\frac{1}{(x^2+1)^2}\,dx=\frac{1}{2}(\arctan(x)+\frac{x}{x^2+1})$, so:

$$\begin{aligned}
I_3 \;&=\; \int\tfrac{1}{(x^2+1)^3}\,dx \;=\; \tfrac{2(3)-3}{2(3-1)}I_2+\tfrac{x}{2(3-1)(x^2+1)^2}\\
&=\; \tfrac{3}{4}\tfrac{1}{2}(\arctan(x)+\tfrac{x}{x^2+1})+\tfrac{1}{4}\tfrac{x}{(x^2+1)^2}\\
&=\; \tfrac{3}{8}\arctan(x)+\tfrac{x(3x^2+5)}{8(x^2+1)^2}\,.
\end{aligned}$$