Reducing to standard trig forms. To find an indefinite integral $\int f(x)\,dx$, we transform it by methods like Substitution and Integration by Parts until we reduce to an integral we recognize from before, a "standard form". In the previous section §7.2, we were able to compute most integrals involving products of trig functions, so these are now standard forms to work toward.

A common type of difficult integral involves forms like $\sqrt{\pm x^2 \pm 1}$. We convert such forms into trigonometric integrals, which at first seems to complicate them. However, we take careful advantage of the Pythagorean identities $\cos^2(\theta)+\sin^2(\theta)=1$ and $\tan^2(\theta)+1=\sec^2(\theta)$, so that the resulting trig formulas simplify to do-able integrals.

Our first example is $\int\sqrt{1-x^2}\,dx$, which computes area under a unit semi-circle. This seems simple enough, but no obvious substitution or integration by parts will simplify it. The expression $\sqrt{1-x^2}$ reminds us of the identity $\sqrt{1-\sin^2(\theta)}=\cos(\theta)$, so imagine our integral was obtained from a more complicated one after a trig substitution $x=\sin(\theta)$: the current variable x is actually a function of a previous variable θ, and $dx=\cos(\theta)\,d\theta$. The previous integral would be:

$$\int\sqrt{1-x^2}\,dx \;=\; \int\sqrt{1-\sin^2(\theta)}\cdot\cos(\theta)\,d\theta, \quad \text{where} \begin{cases} x=\sin(\theta) \\ dx=\cos(\theta)\,d\theta. \end{cases}$$

Now this simplifies to a standard form from §7.2:

$$\int\sqrt{1-\sin^2(\theta)}\cdot\cos(\theta)\,d\theta \;=\; \int\cos^2(\theta)\,d\theta \;=\; \tfrac{1}{2}\theta+\tfrac{1}{4}\sin(2\theta) \;=\; \tfrac{1}{2}\theta+\tfrac{1}{2}\sin(\theta)\cos(\theta)\,.$$

Substituting back the original variable: $\theta=\arcsin(x)$, $\sin(\theta)=x$, $\cos(\theta)=\sqrt{1-x^2}$:*

$$\int\sqrt{1-x^2}\,dx \;=\; \tfrac{1}{2}\arcsin(x)+\tfrac{1}{2}x\sqrt{1-x^2}\,.$$

Let's check the area of the unit circle, i.e. twice the total area under $y=\sqrt{1-x^2}$:

$$2\int_{-1}^{1}\sqrt{1-x^2}\,dx \;=\; \Big[\arcsin(x)+x\sqrt{1-x^2}\,\Big]_{x=-1}^{x=1} \;=\; \arcsin(1)-\arcsin(-1) \;=\; \pi.$$

Integrals with $\sqrt{\pm x^2 \pm a^2}$. We choose a reverse trig substitution depending on the signs in the expression, then use the corresponding Pythagorean identity to obtain a standard trig form. See also the table of integrals further below.

$\sqrt{a^2-x^2}$	$x=a\sin(\theta)$	$dx=a\cos(\theta)\,d\theta$	$\sqrt{a^2-a^2\sin^2(\theta)} \;=\; a\cos(\theta)$
$\sqrt{a^2+x^2}$	$x=a\tan(\theta)$	$dx=a\sec^2(\theta)\,d\theta$	$\sqrt{a^2+a^2\tan^2(\theta)} \;=\; a\sec(\theta)$
$\sqrt{x^2-a^2}$	$x=a\sec(\theta)$	$dx=a\tan(\theta)\sec(\theta)\,d\theta$	$\sqrt{a^2\sec^2(\theta)-a^2} \;=\; a\tan(\theta)$

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*See §6.6 for inverse trig functions like $\arcsin=\sin^{-1}$, and their derivatives.

EXAMPLE: $\int \frac{1}{\sqrt{4-x^2}}\,dx$. Let $x = 2\sin(\theta)$, $dx = 2\cos(\theta)\,d\theta$, $\sqrt{4-(2\sin(\theta))^2} = 2\cos(\theta)$:

$$\int \frac{1}{\sqrt{4-x^2}}\,dx \;=\; \int \frac{1}{2\cos(\theta)}\cdot 2\cos(\theta)\,d\theta \;=\; \theta \;=\; \sin^{-1}(\tfrac{1}{2}x)\,.$$

We could do this more directly by manipulating the integrand to the known derivative of $\arcsin(x)$ by the substitution $u = \frac{1}{2}x$:

$$\int \frac{1}{\sqrt{4-x^2}}\,dx \;=\; \int \frac{1}{\sqrt{1-(\frac{1}{2}x)^2}}\cdot \tfrac{1}{2}\,dx \;=\; \int \frac{1}{\sqrt{1-u^2}}\,du \;=\; \sin^{-1}(u) \;=\; \sin^{-1}(\tfrac{1}{2}x).$$

EXAMPLE: $\int \frac{1}{\sqrt{9x^2+4}}\,dx = \int \frac{1}{\sqrt{(3x)^2+2^2}}\,dx$. Let $3x = 2\tan(\theta)$, $dx = \frac{2}{3}\sec^2(\theta)\,d\theta$, $\sqrt{9x^2+4} = \sqrt{(3x)^2+2^2} = \sqrt{(2\tan(\theta))^2+2^2} = 2\sqrt{\tan^2(\theta)+1} = 2\sec(\theta)$:

$$\int \frac{1}{\sqrt{9x^2+4}}\,dx \;=\; \int \frac{1}{2\sec(\theta)}\cdot \tfrac{2}{3}\sec^2(\theta)\,d\theta \;=\; \tfrac{1}{3}\int \sec(\theta)\,d\theta$$

$$=\; \tfrac{1}{3}\ln\bigl|\tan(\theta)+\sec(\theta)\bigr| \;=\; \tfrac{1}{3}\ln\Bigl|\tfrac{3}{2}x + \tfrac{1}{2}\sqrt{9x^2+4}\Bigr|\,.$$

Alternatively, we could use hyperbolic functions (§6.7), satisfying $\cosh^2(t) = \sinh^2(t)+1$. Thus $x = \frac{2}{3}\sinh(t)$, $dx = \frac{2}{3}\cosh(t)\,dt$ and $\sqrt{9x^2+4} = 2\sqrt{\sinh^2(t)+1} = 2\cosh(t)$ gives:

$$\int \frac{1}{\sqrt{9x^2+4}}\,dx \;=\; \int \frac{1}{2\cosh(t)}\tfrac{2}{3}\cosh(t)\,dt \;=\; \tfrac{1}{3}t$$

$$=\; \tfrac{1}{3}\sinh^{-1}(\tfrac{3}{2}x) \;=\; \tfrac{1}{3}\ln\Bigl(\tfrac{3}{2}x+\sqrt{\tfrac{9}{4}x^2+1}\Bigr)\,.$$

A third method is to substitute $u = \frac{3}{2}x$ to get $\frac{1}{3}\int \frac{1}{\sqrt{u^2+1}}\,du = \frac{1}{3}\sinh^{-1}(u) = \frac{1}{3}\sinh^{-1}(\frac{3}{2}x)$.

EXAMPLE: $\int \frac{\sqrt{x^2-25}}{x}\,dx$; $x = 5\sec(\theta)$, $dx = 5\tan(\theta)\sec(\theta)\,d\theta$, $\sqrt{(5\sec(\theta))^2-25} = 5\tan(\theta)$:

$$\int \frac{\sqrt{x^2-25}}{x}\,dx \;=\; \int \frac{5\tan(\theta)}{5\sec(\theta)}\cdot 5\tan(\theta)\sec(\theta)\,d\theta \;=\; 5\int \tan^2(\theta)\,d\theta$$

$$=\; 5\int \sec^2(\theta)-1\,d\theta \;=\; 5\tan(\theta)-5\theta \;=\; \sqrt{x^2-25} - 5\,\sec^{-1}(\tfrac{1}{5}x)\,.$$

Rewriting $\sec^{-1}(u) = \tan^{-1}\sqrt{u^2-1}$ as in §6.6, this becomes: $\sqrt{x^2-25} - 5\tan^{-1}(\frac{1}{5}\sqrt{x^2-25})\,.$

EXAMPLE: $\int \frac{1}{(x^2-4)^{3/2}}\,dx$; $x = 2\sec(\theta)$, $dx = 2\tan(\theta)\sec(\theta)\,d\theta$, $((2\sec(\theta))^2-4)^{\frac{3}{2}} = 8\tan^3(\theta)$:

$$\int \frac{1}{(x^2-4)^{3/2}}\,dx \;=\; \int \frac{1}{8\tan^3(\theta)}\cdot 2\tan(\theta)\sec(\theta)\,d\theta \;=\; \tfrac{1}{4}\int \sin^{-2}(\theta)\cos(\theta)\,d\theta$$

$$=\; -\tfrac{1}{4}\sin(\theta)^{-1} \;=\; -\tfrac{1}{4}\,\frac{\frac{1}{2}x}{\sqrt{(\frac{1}{2}x)^2-1}} \;=\; -\,\frac{x}{4\sqrt{x^2-4}}\,.$$

Or: $x = 2\cosh(t)$, $x^2-4 = 4\sinh^2(t)$ produces $\frac{1}{4}\int \operatorname{csch}^2(t)\,dt = -\frac{1}{4}\coth(t) = -\frac{\cosh(t)}{4\sinh(t)}$.

EXAMPLE: $\int\sqrt{x^2-1}\,dx$; $x=\cosh(t)$, $dx=\sinh(t)\,dt$, $x^2-1=\cosh^2(t)-1=\sinh^2(t)$:

$$\begin{aligned}\int\sqrt{x^2-1}\,dx &= \int\sqrt{\cosh^2(t)-1}\ \sinh(t)\,dt\\ &= \int\sinh^2(t)\,dt \;=\; \tfrac12\cosh(t)\sinh(t)-\tfrac12 t\\ &= \tfrac12 x\sqrt{x^2-1}-\tfrac12\cosh^{-1}(x),\\ &= \tfrac12 x\sqrt{x^2-1}-\tfrac12\ln(x+\sqrt{x^2-1}),\end{aligned}$$

using the formulas for $\int\sinh^2(t)\,dt$ and $\cosh^{-1}(x)$ from §6.7.

EXAMPLE: $\int\sqrt{1-x^2}\,dx$; $x=\tanh(t)$, $dx=\operatorname{sech}^2(t)\,dt$, $1-x^2=1-\tanh^2(t)=\operatorname{sech}^2(t)$:

$$\begin{aligned}\int\sqrt{1-x^2}\,dx &= \int\operatorname{sech}^3(t)\,dt \;=\; \frac12\left(\tanh(t)\operatorname{sech}(t)+\int\operatorname{sech}(t)\,dt\right)\\ &= \tfrac12\tanh(t)\operatorname{sech}(t)+\tan^{-1}(e^t) \;=\; \tfrac12 x\sqrt{1-x^2}+\tan^{-1}\sqrt{\tfrac{1+x}{1-x}},\end{aligned}$$

via §6.7, 7.2. This simplifies to the previous $\int\sqrt{1-x^2}\,dx \;=\; \frac12 x\sqrt{1-x^2}+\frac12\sin^{-1}(x)+\frac{\pi}{4}$.

EXAMPLE: $\int\frac{1}{x\sqrt{x^2+x}}\,dx$. It will not help to complete the square and do a linear substitution $u=ax+b$, since the denominator will still contain the factor $x=\frac1a u-b$. Rather, write $x^2+x=x^2(1+\frac1x)$ and do a linear fractional substitution $u=1+\frac1x$, $du=-\frac{1}{x^2}\,dx$:

$$\int\frac{1}{x\sqrt{x^2+x}}\,dx \;=\; \int\frac{1}{x^2\sqrt{1+\frac1x}}\,dx \;=\; -\int\frac{1}{\sqrt u}\,du \;=\; -2\sqrt u \;=\; -2\sqrt{1+\tfrac1x}.$$

EXAMPLE: $\int\frac{1}{x\sqrt{3x^2-2x-1}}\,dx$. Using the strategy of the previous example, we rewrite as a quadratic in $\frac1x$ and complete the square in terms of $u=\frac ax+b$:

$$\begin{aligned}3x^2-2x-1 &= x^2(3-\tfrac2x-\tfrac{1}{x^2})\\ &= x^2\left(3+1^2-1^2-2(1)\tfrac1x-\tfrac{1}{x^2}\right)\\ &= x^2\left(4-(1+\tfrac1x)^2\right) \;=\; 4x^2\left(1-(\tfrac12+\tfrac{1}{2x})^2\right).\end{aligned}$$

Then $u=\frac{1}{2x}+\frac12$, $du=-\frac{1}{2x^2}\,dx$ gives:

$$\int\frac{1}{x\sqrt{3x^2-2x-1}}\,dx \;=\; \int\frac{1}{2x^2\sqrt{1-(\frac12+\frac{1}{2x})^2}}\,dx \;=\; \int\frac{-1}{\sqrt{1-u^2}}\,du \;=\; -\sin^{-1}(u) \;=\; -\sin^{-1}(\tfrac{1}{2x}+\tfrac12).$$

EXAMPLE: $\int\sqrt{\frac{x}{x-1}}\,dx$. It turns out best to substitute $u=\sqrt{x-1}$, $x=u^2+1$, $dx=2u\,du$:

$$\int\sqrt{\frac{x}{x-1}}\,dx \;=\; \int\frac{\sqrt{u^2+1}}{u}\,2u\,du \;=\; \int 2\sqrt{u^2+1}\,du \;=\; u\sqrt{u^2+1}+\sinh^{-1}(u)$$

$$= \sqrt{x(x-1)}+\sinh^{-1}\sqrt{x-1} \;=\; \sqrt{x(x-1)}+\ln(\sqrt x+\sqrt{x-1})$$

EXAMPLE: $\int \frac{1}{\sqrt{x^2+x}}\,dx$. In §6.7, completing the square gave $\cosh^{-1}(2x{+}1) = \ln(2x{+}1{+}2\sqrt{x^2{+}x})$, but sneaky substitution gives a surprise equivalent answer: $z=\sqrt{x},\ x=z^2,\ dx=2z\,dz$;

$$\int \frac{1}{\sqrt{x^2+x}}\,dx \;=\; \int \frac{1}{\sqrt{z^4+z^2}}\,2z\,dz \;=\; \int \frac{2}{\sqrt{z^2+1}}\,dz \;=\; 2\sinh^{-1}(z) \;=\; 2\sinh^{-1}\!\sqrt{x}\,.$$

EXAMPLE: $\int \sec^3(x)\,dx$. We did this notorious example in §7.2 via integration by parts, but double trig-hyperbolic substitution also works. We have $\sec(x) = \sqrt{1{+}\tan^2(x)}$, which is similar to the hyperbolic identity $\cosh(t) = \sqrt{1{+}\sinh^2(t)}$. Thus, we substitute:

$$\tan(x) \;=\; \sinh(t), \qquad \sec^2(x)\,dx = \cosh(t)\,dt,$$

$$\int \sec^3(x)\,dx \;=\; \int \sqrt{1{+}\tan^2(x)}\,\sec^2(x)\,dx \;=\; \int \sqrt{1{+}\sinh^2(t)}\,\cosh(t)\,dt$$

$$=\; \int \cosh^2(t)\,dt \;=\; \int \left(\tfrac{1}{2}\cosh(2t){+}\tfrac{1}{2}\right)dt \;=\; \tfrac{1}{4}\sinh(2t) + \tfrac{1}{2}t$$

$$=\; \tfrac{1}{2}\sinh(t)\cosh(t) + \tfrac{1}{2}t \;=\; \tfrac{1}{2}\tan(x)\sec(x) + \tfrac{1}{2}\sinh^{-1}(\tan(x)).$$

Then $\sinh^{-1}(x) = \ln(x+\sqrt{1{+}x^2})$ recovers the previous answer:

$$\int \sec^3(x)\,dx \;=\; \tfrac{1}{2}\tan(x)\sec(x) + \tfrac{1}{2}\ln(\tan(x){+}\sec(x)).$$

Table of integrals which produce inverse trig and hyperbolic functions (omitting $+C$).

$$\int \frac{1}{\sqrt{1-x^2}}\,dx \;=\; \sin^{-1}(x) \;=\; \tfrac{\pi}{2}{-}\cos^{-1}(x) \qquad \int \frac{1}{\sqrt{x^2-1}}\,dx \;=\; \cosh^{-1}(x) \;=\; \ln(x{+}\sqrt{x^2-1})$$

$$\int \frac{1}{\sqrt{1+x^2}}\,dx \;=\; \sinh^{-1}(x) \;=\; \ln(x+\sqrt{1+x^2})$$

$$\int \frac{1}{x\sqrt{x^2{-}1}}\,dx \;=\; \sec^{-1}(x) \;=\; \tan^{-1}\!\sqrt{x^2{-}1}\,^{\dagger} \qquad \int \frac{1}{x\sqrt{1{-}x^2}}\,dx \;=\; -\mathrm{sech}^{-1}(x) \;=\; \ln(x) - \ln(1{+}\sqrt{1{-}x^2})$$

$$\int \frac{1}{x\sqrt{1+x^2}}\,dx \;=\; -\mathrm{csch}^{-1}(x) \;=\; \ln(x) - \ln(1+\sqrt{1+x^2})$$

$$\int \frac{1}{1+x^2}\,dx \;=\; \tan^{-1}(x) \qquad \int \frac{1}{1-x^2}\,dx \;=\; \tanh^{-1}(x) \;=\; \tfrac{1}{2}\ln(1+x) - \tfrac{1}{2}\ln(1-x)$$

$$\int \sqrt{1-x^2}\,dx \;=\; \tfrac{1}{2}x\sqrt{1-x^2} + \tfrac{1}{2}\sin^{-1}(x) \qquad \int \sqrt{x^2-1}\,dx \;=\; \tfrac{1}{2}x\sqrt{x^2-1} - \tfrac{1}{2}\cosh^{-1}(x)$$

$$\int \sqrt{x^2+1}\,dx \;=\; \tfrac{1}{2}x\sqrt{x^2+1} + \tfrac{1}{2}\sinh^{-1}(x)$$

† Equality holds for $x \ge 1$; for $x \le -1$, see end of §6.6.