Products by substitution. In this section, we develop methods to find indefinite integrals (antiderivatives) of products of trig functions, beyond the direct antiderivatives:

$$\int \cos(x)\,dx = \sin(x), \quad \int \sin(x)\,dx = -\cos(x), \quad \int \sec^2(x)\,dx = \tan(x),$$

$$\int \tan(x)\sec(x)\,dx = \sec(x), \quad \int \csc^2(x)\,dx = -\cot(x), \quad \int \cot(x)\csc(x)\,dx = -\csc(x).$$

The easiest cases allow a simple trig substitution reducing to a polynomial, often using $\sin^2 + \cos^2 = 1$ or $\tan^2 + 1 = \sec^2$. Let n, m be integers, with $n \geq 0$.

(a) $\int \cos^{2n+1}(x)\sin^m(x)\,dx$: take $u = \sin(x)$, $du = \cos(x)\,dx$.
$\int \sin^{2n+1}(x)\cos^m(x)\,dx$: take $u = \cos(x)$, $du = -\sin(x)\,dx$.

$$\begin{aligned}
\int \cos^3(x)\sin^{-10}(x)\,dx &= \int \cos^2(x)\sin^{-10}(x)\cdot\cos(x)\,dx \\
&= \int (1-\sin^2(x))\sin^{-10}(x)\cdot\cos(x)\,dx \\
&= \int (1-u^2)u^{-10}\,du \\
&= \int u^{-10} - u^{-8}\,du \\
&= \tfrac{1}{-9}u^{-9} - \tfrac{1}{-7}u^{-7} + C \\
&= -\tfrac{1}{9}\sin^{-9}(x) + \tfrac{1}{7}\sin^{-7}(x) + C.
\end{aligned}$$

$$\begin{aligned}
\int \sin^5(x)\,dx &= -\int \sin^4(x)\cdot(-\sin(x))\,dx \\
&= -\int (1-\cos^2(x))^2\cdot(-\sin(x))\,dx \\
&= -\int (1-u^2)^2\,du \\
&= -\int (1-2u^2+u^4)\,du \\
&= -u + \tfrac{2}{3}u^3 - \tfrac{1}{5}u^5 + C \\
&= -\cos(x) + \tfrac{2}{3}\cos^3(x) - \tfrac{1}{5}\cos^5(x) + C.
\end{aligned}$$

(b) $\int \sec^{2n+2}(x)\tan^m(x)\,dx$: take $u = \tan(x)$, $du = \sec^2(x)\,dx$.

$$\begin{aligned}
\int \sec^6(x)\tan^{-3}(x)\,dx &= \int \sec^4(x)\tan^{-3}(x)\cdot\sec^2(x)\,dx \\
&= \int (\tan^2(x)+1)^2\tan^{-3}(x)\cdot\sec^2(x)\,du \\
&= \int (u^2+1)^2\,u^{-3}\,du \\
&= \int (u + 2u^{-1} + u^{-3})\,du \\
&= \tfrac{1}{2}u^2 + \ln|u| - \tfrac{1}{2}u^{-2} + C \\
&= \tfrac{1}{2}\tan^2(x) + \ln|\tan(x)| - \tfrac{1}{2}\cot^2(x) + C.
\end{aligned}$$

For brevity, we will henceforth omit the constant of integration $+C$.

Remaining cases. What if a product of trig functions does not fit types (a) or (b)?

- Rewrite in terms of sin and cos, obtaining $\sin^p(x)\cos^q(x)$ for integers p, q. Use type (a) if p is odd and positive, or if q is odd and positive. Reduce to type (b) if p and q are both even with $p+q$ negative; or if p and q are both odd and negative.

$$\int \sin^2(x)\cos(x)\tan^2(x)\csc(x)\,dx \;=\; \int \sin^2(x)\cos(x)\,\frac{\sin^2(x)}{\cos^2(x)}\,\frac{1}{\sin(x)}\,dx$$

$$= \int \sin^3(x)(\cos(x))^{-1}\,dx \;=\; -\ln|\cos(x)| + \tfrac{1}{2}\cos^2(x), \quad \text{type (a).}$$

$$\int \csc^2(x)\,dx \;=\; \int \frac{1}{\sin^2(x)}\,dx \;=\; \int \frac{1}{\cos^2(x)}\,\frac{\cos^2(x)}{\sin^2(x)}\,dx \;=\; \int \sec^2(x)\tan^{-2}(x)\,dx$$

$$= \int u^{-2}\,du \;=\; -u^{-1} \;=\; = \;-(\tan(x))^{-1} \;=\; -\cot(x), \quad \text{type (b).}$$

$$\int \frac{1}{\sin^3(x)\cos(x)}\,dx \;=\; \int \frac{1}{\cos^4(x)}\,\frac{\cos^3(x)}{\sin^3(x)}\,dx \;=\; \int \sec^4(x)\tan^{-3}(x)\,dx$$

$$= \int (u^2+1)u^{-3}\,du \;=\; \ln|\tan(x)| - \tfrac{1}{2}\cot^2(x), \quad \text{type (b).}$$

- Pythagorean identities: $\int \tan^4(x)\,dx \;=\; \int(\sec^2(x)-1)\tan^2(x)\,dx$ $=\; \int \sec^2(x)\tan^2(x)\,dx - \int(\sec^2(x)-1)\,dx \;=\; \tfrac{1}{3}\tan^3(x) - \tan(x) + x$, type (b).

- For even positive powers of sin and cos, rewrite using the identities: $\sin^2(x) = \tfrac{1}{2}(1-\cos(2x))$, $\cos^2(x) = \tfrac{1}{2}(1+\cos(2x))$, $\sin(x)\cos(x) = \tfrac{1}{2}\sin(2x)$.

$$\int \sin^6(x)\,dx \;=\; \int (\tfrac{1}{2}(1-\cos(2x))^3\,dx$$

$$= \tfrac{1}{8}\int 1 - 3\cos(2x) + 3\cos^2(2x) - \cos^3(2x)\;dx$$

$$= \tfrac{1}{8}\int 1 - 3\cos(2x) + \tfrac{3}{2}(1+\cos(4x)) - \cos^3(2x)\;dx\,,$$

  where we used the binomial formula $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Now each term can be done directly, or by type (a) with the substitution $u = \sin(2x)$.

Recalcitrant cases. In case of odd negative powers of sin or cos, we need special tricks.

EXAMPLE: The integral of secant $\int \sec(x)\,dx$ was needed by map-makers in the 1600's, when Calculus was first developed. It calibrates stretching in the Mercator projection, which makes map directions match compass directions. An amazing trick from §6.2:

$$\int \sec(x)\,dx \;=\; \int \sec(x)\,\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx$$

$$= \int \frac{1}{\sec(x)+\tan(x)}\cdot\big(\sec^2(x)+\sec(x)\tan(x)\big)\,dx \;=\; \int \frac{1}{u}\,du$$

for $u = \sec(x)+\tan(x)$, $du = (\sec(x)\tan(x)+\sec^2(x))\,dx$.

Thus:

$$\int \sec(x)\,dx \;=\; \ln|u| \;=\; \ln|\sec(x){+}\tan(x)|.$$

Another trick for this is to write $\int\sec(x)\,dx = \int \frac{1}{\cos^2(x)}\cos(x)\,dx$, and substitute $u = \sin(x)$ to get $\int\frac{1}{1-u^2}\,du$. We will see how to integrate such rational functions in §7.4.

EXAMPLE: Here is a tricky integration by parts, in which we get back to the same integral we started with:

$$\int \sec^3(x)\,dx \;=\; \int \sec(x)(1+\tan^2(x))\,dx \;=\; \int \sec(x)\,dx + \int \underbrace{\tan(x)}_{u}\,\underbrace{\sec(x)\tan(x)\,dx}_{dv}$$

$$=\; \ln|\sec(x){+}\tan(x)| + \underbrace{\tan(x)}_{u}\,\underbrace{\sec(x)}_{v} - \int \underbrace{\sec(x)}_{v}\,\underbrace{\sec^2(x)\,dx}_{du}$$

Since we have $\int\sec^3(x)\,dx$ on both sides *with opposite signs*, we can solve for it to get:

$$\int \sec^3(x)\,dx \;=\; \tfrac{1}{2}\Big(\ln\big|\sec(x){+}\tan(x)\big| \;+\; \tan(x)\sec(x)\Big).$$

Trig integrals with inside coefficients. Add together the angle addition identities:

$$\cos(a{+}b) \;=\; \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\cos(a{-}b) \;=\; \cos(a)\cos(b) + \sin(a)\sin(b)$$

$$\tfrac{1}{2}(\cos(a{+}b) + \cos(a{-}b)) \;=\; \cos(a)\cos(b).$$

This allows us to do integrals of the form:

$$\int \cos(nx)\cos(mx)\,dx \;=\; \int \tfrac{1}{2}\cos(nx{+}mx) + \tfrac{1}{2}\cos(nx{-}mx)\;dx$$

$$=\; \tfrac{1}{2(n+m)}\sin\big((n{+}m)x\big) + \tfrac{1}{2(n-m)}\sin\big((n{-}m)x\big) + C.$$

Similarly:

$$\tfrac{1}{2}(\cos(a{-}b) - \cos(a{+}b)) \;=\; \sin(a)\sin(b)$$
$$\tfrac{1}{2}(\sin(a{+}b) + \sin(a{-}b)) \;=\; \sin(a)\cos(b).$$

Tangent half-angle substitution. For a really messy trigonometric integral like:

$$\int \frac{\cos^2(x)\sin(x) - 2\tan(x) + 1}{\sec^3(x) + \sin^3(x) + 3\cos(x)\sin(x) + 5}\;dx,$$

there is a general method in the extra notes, the Geometric Trig Substitution:

$$\sin(x) = \frac{2t}{1+t^2}\,, \qquad \cos(x) = \frac{1-t^2}{1+t^2}\,, \qquad dx = \frac{2}{1+t^2}\,dt\,, \qquad t = \tan(\tfrac{1}{2}x) = \frac{\sin(x)}{1+\cos(x)}\,.$$

This reduces the trig integral to the integral of a rational function, which can be done by Partial Fractions (§7.4).