Integration by Parts

Review of integrals. The definite integral gives the cumulative total of many small parts, such as the slivers which add up to the area under a graph. Numerically, it is a limit of Riemann sums:

$$\int_a^b f(x)\,dx \;=\; \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x\,,$$

where we divide the interval $x \in [a,b]$ into n increments of size $\Delta x = \frac{b-a}{n}$ with division points $a < a{+}\Delta x < a{+}2\Delta x < \cdots < a{+}n\Delta x = b$, and $x_1, \ldots x_n$ are sample points from each increment. This definition is not a theoretical curiosity: it is the reason integrals are relevant to physical problems, and it is the only way to evaluate most integrals: there is no algebraic way.

However, for sufficiently simple functions $f(x)$, we can evaluate integrals algebraically by the shortcut of the Second Fundamental Theorem of Calculus. This says that if $f(x)$ is the rate of change of some known antiderivative $F(x)$, then the integral of $f(x)$ is the cumulative total change of $F(x)$:

$$F'(x) = f(x) \qquad \Longrightarrow \qquad \int_a^b f(x)\,dx \;=\; F(x)\big|_{x=a}^{x=b} \;=\; F(b) - F(a)\,.$$

(The First Fundamental Theorem says that the definite integral gives an antiderivative even if there is no formula $F(x)$: defining $I(x) = \int_a^x f(t)\,dt$, we have $I'(x) = f(x)$.)

Algebraic integration is the process of finding antiderivative formulas, denoted as indefinite integrals $\int f(x)\,dx = F(x) + C$. The most direct method is to reverse Basic Derivatives, such as $(x^p)' = px^{p-1}$ reversing to $\int x^p\,dx = \frac{x^{p+1}}{p+1}$. Our only other integration method so far is the Substitution Method, which reverses the Chain Rule:

$$\int f(g(x))\,g'(x)\,dx \;=\; \int f(u)\,du \;=\; F(u) + C \quad \text{where } u = g(x) \text{ and } F'(u) = f(u).$$

Reversing the Product Rule. Since we have:

$$(f(x)\,g(x))' \;=\; f(x)\,g'(x) + g(x)\,f'(x)\,,$$

we can take the antiderivative of both sides to give:

$$f(x)\,g(x) \;=\; \int f(x)\,g'(x)\,dx + \int g(x)\,f'(x)\,dx\,,$$

$$\int f(x)\,g'(x)\,dx \;=\; f(x)\,g(x) - \int g(x)\,f'(x)\,dx\,.$$

In Leibniz notation, taking $u = f(x)$, $du = f'(x)\,dx$ and $v = g(x)$, $dv = g'(x)\,dx$:

$$\int u\;dv \;=\; uv \;-\; \int v\;du\,.$$

This method transforms the integral of a product $f(x)\,g'(x)$ into $f(x)\,g(x)$ minus the integral of $g(x)\,f'(x)$, the other term in the Product Rule; we can think of lowering $f(x)$ to its derivative $f'(x)$ and raising $g'(x)$ to its antiderivative $g(x)$.

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Method for Integration by Parts.

1. Given an indefinite integral $\int h(x)\,dx$, find a factor of the integrand $h(x)$ which you recognize as the derivative of a function $g(x)$: that is, write $h(x) = f(x)\cdot g'(x)$.

2. Taking $u = f(x)$, $dv = g'(x)\,dx$, transform the integral $\int h(x)\,dx = \int u\,dv$ into $uv - \int v\,du \;=\; f(x)g(x) - \int g(x)\,f'(x)\,dx$.

3. Simplify $g(x)\,f'(x)$, possibly using identities, and try to find its integral by other methods such as Substitution.

4. Sometimes you can repeat Steps 1 & 2 on $\int g(x)\,f'(x)\,dx$ with a *different* u, v.* This might result in a simpler integral which you can evaluate by other methods.

5. Instead of simplifying the integral, Step 3 or 4 might give an expression with the same integral you started with. Solve the resulting equation to find that integral.

Notice that Step 1 is the same as for the Method of Substitution, where you must find a factor of the integrand which is a known derivative $g'(x)$; but for Substitution, $g(x)$ must also appear as an inside function in the remaining factor: $h(x) = f(g(x))\cdot g'(x)$.

EXAMPLE: Evaluate $\int x\cos(x)\,dx$. There are two obvious candidates for u, v. First, if we take $u = \cos(x)$, $dv = x\,dx$, we get $du = -\sin(x)\,dx$, $v = \frac{1}{2}x^2$, and:

$$\begin{array}{rcccl} \int u\,dv & = & uv & - & \int v\,du \\ \int \cos(x)\,x\,dx & = & \cos(x)\left(\frac{1}{2}x^2\right) & - & \int \frac{1}{2}x^2\,(-\sin(x))\,dx \end{array}$$

Unfortunately, the new integral $\int x^2\sin(x)\,dx$ is harder than the original $\int x\cos(x)\,dx$. We must make a wiser choice of u, v, so that the derivative du will be simpler than the original u, while the antiderivative v will be no worse than the original dv.

The other obvious choice will work: take $u = x$, $dv = \cos(x)\,dx$, so that $du = 1\,dx$ and $v = \sin(x)$. Then:†

$$\begin{array}{rcccl} \int u\,dv & = & uv & - & \int v\,du \\ \int x\,\cos(x)\,dx & = & x\,\sin(x) & - & \int \sin(x)\,1\,dx \\ & = & x\sin(x) & + & \cos(x). \end{array}$$

Thus, Steps 1–3 were enough to integrate.

To check our answer, we reverse our Integration by Parts using the Product Rule:

$$(x\sin(x)+\cos(x))' \;=\; x\sin'(x)+\sin(x)(x)'+\cos'(x)$$

$$=\; x\cos(x)+\sin(x)-\sin(x) \;=\; x\cos(x).$$

*Repeating with the *same* factorization $\int v\,du$ would get back the original integral $\int u\,dv$.

†For brevity, we again neglect the arbitrary constant $+C$ in a general antiderivative, though you should write it on a test or quiz.

EXAMPLE: Evaluate $\int x^2 e^{-x}\,dx$. We should choose $u = x^2$, $dv = e^{-x}$, so that $du = 2x\,dx$ is simpler, but $v = -e^{-x}$ is no more complicated:

$$\begin{array}{rcccc} \int u\,dv & = & uv & - & \int v\,du \\ \int x^2 e^{-x}\,dx & = & x^2(-e^{-x}) & - & \int (-e^{-x})\,2x\,dx \\ & = & -x^2 e^{-x} & + & 2\int x e^{-x}\,dx \end{array}$$

Going on to Step 4, we repeat the process for the integral on the right side, this time with $u = x$, $dv = e^{-x}\,dx$ and $du = dx$, $v = -e^{-x}$:

$$\begin{array}{rcccc} \int u\,dv & = & uv & - & \int v\,du \\ \int x\,e^{-x}\,dx & = & x(-e^{-x}) & - & \int (-e^{-x})\,dx \\ & = & -xe^{-x} & + & \int e^{-x}\,dx \\ & = & -xe^{-x} & + & (-e^{-x}) \end{array}$$

Putting these together:

$$\int x^2\,e^{-x}\,dx \;=\; -x^2 e^{-x} + 2(-xe^{-x} - e^{-x}) \;=\; -(x^2 + 2x + 2)e^{-x}.$$

EXAMPLE: Evaluate $\int e^x \sin(x)\,dx$. Steps 1–4 give:

$$\begin{array}{rcclr} \int e^x \sin(x)\,dx & = & e^x \sin(x) & - \;\; \int e^x \cos(x)\,dx, & u = \sin(x), v = e^x \\ & = & e^x \sin(x) & - \;\; \left(e^x \cos(x) - \int e^x(-\sin(x))\,dx\right), & u = \cos(x), v = e^x. \end{array}$$

We conclude:

$$\int e^x \sin(x)\,dx \;=\; e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x)\,dx.$$

Since our integral $\int e^x \sin(x)\,dx$ appears on both sides, we go to Step 5 and solve for it:

$$\int e^x \sin(x)\,dx \;=\; \tfrac{1}{2}\left(e^x \sin(x) - e^x \cos(x)\right).$$

EXAMPLE: Evaluate $\int \ln(x)\,dx$. Here there does not seem to be any dv factor, but we can always take $dv = 1\,dx$, so $v = x$:

$$\begin{array}{rcccc} \int u\,dv & = & uv & - & \int v\,du \\ \int \ln(x)\,1\,dx & = & \ln(x)\,x & - & \int x\,\frac{1}{x}\,dx \\ & = & x\ln(x) & - & x. \end{array}$$

EXAMPLE: Evaluate $\int \sin^{-1}(x)\,dx$.[‡] Again we must use $u = \sin^{-1}(x)$ and $dv = 1\,dx$, counting on the fact that du is simpler than u:

$$\begin{array}{rcccc} \int u\,dv & = & uv & - & \int v\,du \\ \int \sin^{-1}(x)\,1\,dx & = & \sin^{-1}(x)\,x & - & \int x\,\frac{1}{\sqrt{1-x^2}}\,dx \end{array}$$

Continuing Step 3, we use the substitution $z = 1 - x^2$ on the right-hand integral:

$$\int x\,\frac{1}{\sqrt{1-x^2}}\,dx \;=\; -\tfrac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,(-2x)\,dx \;=\; -\tfrac{1}{2}\int \frac{1}{\sqrt{z}}\,dz \;=\; -\sqrt{z} \;=\; -\sqrt{1-x^2}.$$

Combining:

$$\int \sin^{-1}(x)\,dx \;=\; x\sin^{-1}(x) - (-\sqrt{1-x^2}) \;=\; x\sin^{-1}(x) + \sqrt{1-x^2}.$$

[‡]Notation: $\sin^{-1}(x) = \arcsin(x)$, but $\sin(x)^{-1} = \frac{1}{\sin(x)} = \csc(x)$.