Method for Integration

Given a function $f(x)$, we wish to find the indefinite integral $\int f(x)\,dx = F(x)+C$, i.e. an antiderivative function with $F'(x) = f(x)$. For brevity, we omit the constant $+C$.

1. Basic integrals which directly reverse basic derivatives:

$$\int x^p\,dx = \tfrac{1}{p+1}x^{p+1}\ (p\neq -1) \qquad \int \tfrac{1}{x}\,dx = \ln|x| \qquad \int e^x\,dx = e^x$$

$$\int \sin(x)\,dx = -\cos(x) \qquad \int \cos(x)\,dx = \sin(x)$$

$$\int \sec^2(x)\,dx = \tan(x) \qquad \int \tan(x)\sec(x)\,dx = \sec(x)$$

$$\int \tfrac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) \qquad \int \tfrac{1}{1+x^2}\,dx = \arctan(x)$$

2. Substitution: Factor the integrand so that $\int f(x)\,dx = \int h(g(x))\cdot g'(x)\,dx$.

   Take $u = g(x)$, $du = g'(x)\,dx$, so that $\int h(g(x))\,g'(x)\,dx = \int h(u)\,du$. Integrate to get $\int h(u)\,du = H(u)$. Restore the original variable: $\int f(x)\,dx = H(g(x))$.

   Tips: Take an inside function $u = g(x)$; if there is no factor $du = g'(x)\,dx$, multiply by $\frac{1}{g'(x)}g'(x)$, or take inverse function $x = g^{-1}(u)$, $dx = (g^{-1})'(u)\,du$. Or start with a factor $du = g'(x)\,dx$; and in the other factor, reverse-substitute $x = g^{-1}(u)$.

3. Integration by Parts. Find a known derivative $g'(x)$ as a factor of the integrand: $\int f(x)\,dx = \int h(x)\cdot g'(x)\,dx = h(x)\cdot g(x) - \int g(x)\cdot h'(x)\,dx$, i.e. $\int u\,dv = uv - \int v\,du$.

   The remaining integral $\int g(x)\cdot h'(x)\,dx$ should be easier, provided $h'(x)$ is *simpler* than $h(x)$, but $g(x)$ is *about as complicated* as $g'(x)$. Or try $g'(x) = 1$, $g(x) = x$.

   After identities, the right side may again contain $-\int f(x)\,dx$: solve for the integral.

4. Products of Trig Functions. Substitute by factoring out a derivative $g'(x) = \cos(x)$, $\sin(x)$, $\sec^2(x)$ or $\tan(x)\sec(x)$; write remaining factor in terms of $u = g(x)$ using $\cos^2(x)+\sin^2(x) = 1$, $\tan^2(x)+1 = \sec^2(x)$, $\tan(x) = \frac{\sin(x)}{\cos(x)}$, $\sec(x) = \frac{1}{\cos(x)}$.

   Otherwise, use identities $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$, $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$.

   A hard case: $\int \sec(x)\,dx = \ln|\tan(x)+\sec(x)|$. Geometric substitution converts any trig integral to rational: $\cos(\theta) = \frac{1-t^2}{1+t^2}$, $\sin(\theta) = \frac{2t}{1+t^2}$, $d\theta = \frac{2}{1+t^2}\,dt$, $t = \tan(\frac{1}{2}\theta)$.

5. Reverse Trig Substitution. If $\sqrt{a^2-x^2}$ appears in $\int f(x)\,dx$, complicate it by substituting $x = a\sin(\theta)$, $dx = a\cos(\theta)\,d\theta$; simplify $\sqrt{a^2-x^2} = \sqrt{a^2-(a\sin(\theta))^2} = a\cos(\theta)$. Do the resulting trig integral; then restore $\theta = \sin^{-1}(\frac{x}{a})$, $\sin(\theta) = \frac{x}{a}$, $\cos(\theta) = \frac{1}{a}\sqrt{a^2-x^2}$.

   For $\sqrt{x^2-a^2}$, use $x = a\sec(\theta)$ or $a\cosh(t)$; for $\sqrt{x^2+a^2}$, use $x = a\tan(\theta)$ or $a\sinh(t)$.

6. Partial Fractions integrates rational functions $f(x) = \frac{g(x)}{h(x)} = \frac{\text{polynomial}}{\text{polynomial}}$. If $g(x)$ has degree greater than or equal to $h(x)$, perform long division to get $f(x) = q(x) + \frac{r(x)}{h(x)}$, where $r(x)$ has degree less than $h(x)$, and proceed with $\frac{r(x)}{h(x)}$.

   If the denominator factors as $h(x) = (x-a)(x-b)\cdots$ with $a, b, \ldots$ all different, split $f(x)$ into the form: $f(x) = \frac{g(x)}{(x-a)(x-b)\cdots} = \frac{A}{x-a} + \frac{B}{x-b} + \cdots$. Solve for the constant A after clearing denominators and substituting $x = a$; substitute $x = b$ to solve for B, etc. Finally, integrate using $\int \frac{A}{x-a}\,dx = A\ln|x-a| = \ln|x-a|^A$.

   Complete the square in denom: $ax^2+bx+c = a(x+k)^2+\ell$ with $k = \frac{b}{2a}$, $\ell = -(\frac{b}{2a})^2 + c$. For $a, \ell > 0$: $\int \frac{(x+k)}{a(x+k)^2+\ell}\,dx = \frac{1}{2a}\ln(ax^2+bx+c)$; $\int \frac{1}{a(x+k)^2+\ell}dx = \frac{1}{\sqrt{a\ell}}\arctan\sqrt{\frac{a}{\ell}}(x+k)$.

   See §7.4 for higher terms if $h(x)$ has factors $(x-a)^n$ or irreducible $(ax^2+bx+c)^n$.

7. An integral has no elementary formula if it reduces to one of these special functions: sine integral $\mathrm{Si}(x) = \int \frac{\sin(x)}{x}\,dx$, but Dirichlet $\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx = \pi$; exp int $\mathrm{Ei}(x) = \int \frac{e^x}{x}\,dx$; error function $\sqrt{\pi}\,\mathrm{erf}(x) = \int e^{-x^2}dx$, but Gauss $\int_{-\infty}^{\infty} e^{-x^2}dx = \Gamma(\frac{1}{2}) = \sqrt{\pi}$; log int $\mathrm{Li}(x) = \int \frac{1}{\ln(x)}\,dx$, but Feynman $\int_0^1 \frac{x^p-1}{\ln(x)}\,dx = \ln(\frac{1}{p+1})$; dilog $\mathrm{Li}_2(x) = -\int \frac{\ln(1-x)}{x}\,dx$; elliptic integrals, $k \neq 1$: $\int \frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}}$, $\int \sqrt{1-k^2\sin^2(\theta)}\,d\theta$, $\int \frac{d\theta}{(1-n\sin^2(\theta))\sqrt{1-k^2\sin^2(\theta)}}$; gamma $\Gamma(z) = (z-1)! = \int_0^\infty t^{z-1}e^{-zt}\,dt$; beta $B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)} = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}dt$.