Math 133

Hyperbolic Functions

Stewart §6.7

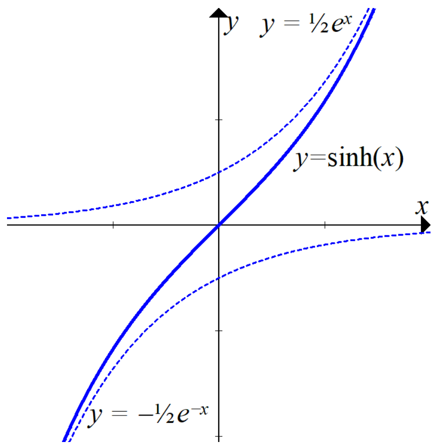
Definitions. Besides the *algebraic functions* defined by arithmetic operations, constant powers, and roots, we have seen several types of *transcendental functions* such as e^x, the trigonometric functions, and their inverse functions. Now we introduce the *hyperbolic functions*, a new class of transcendental functions which appear in some scientific and mathematical applications (though much less commonly than our previous functions).

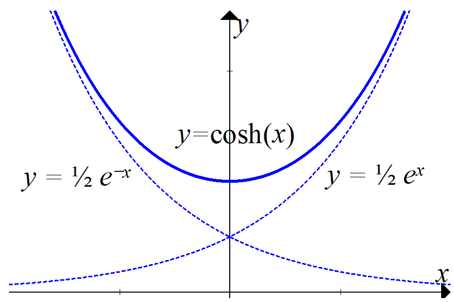
Each hyperbolic function corresponds to a trigonometric function: to the ordinary sine function $\sin(x)$ there corresponds the hyperbolic sine, written $\sinh(x)$; to the ordinary tangent there corresponds the hyperbolic tangent $\tanh(x)$, etc.* These new functions are defined in terms of exponential functions:

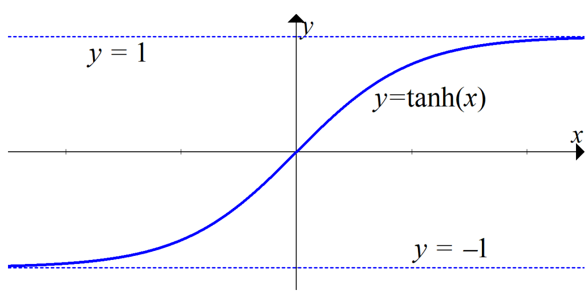
$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} \qquad \operatorname{csch}(x) = \frac{1}{\sinh(x)} \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)}$$

That is, $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, etc. Graphs are easy to picture from $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$:







Notice that $\sinh(x)$ is an odd function like $\sin(x)$, meaning $f(-x) = -f(x)$; and $\cosh(x)$ is an even function like $\cos(x)$, meaning $f(-x) = f(x)$. Also, $e^x = \sinh(x) + \cosh(x)$, so the two primary hyperbolic functions are the odd and even components of the exponential function.†

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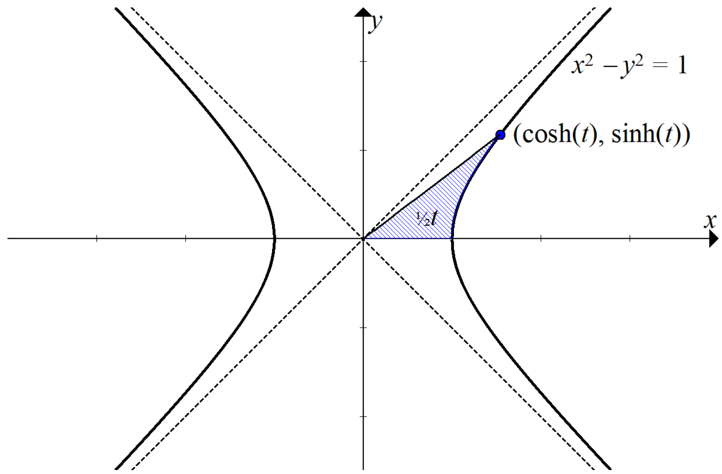
*We pronounce sinh as "sinch", cosh as "kosh", tanh as "tanch", etc.

†The hyperbolic $e^x = \cosh(x) + \sinh(x)$ corresponds to Euler's formula $e^{ix} = \cos(x) + i\sin(x)$, where $i = \sqrt{-1}$. Comparing, we find $\cosh(x) = \cos(ix)$ and $\sinh(x) = -i\sin(ix)$, which explains the analogy.

Geometric meaning. Why the trigonometric nomenclature? The most important geometric role of the trigonometric functions is to pamametrize circular motion: $(x, y) = (\cos(t), \sin(t))$ traces out the unit circle for $t \in [0, 2\pi]$. This is because the circle equation $x^2 + y^2 = 1$ corresponds to the identity $\cos^2(t) + \sin^2(t) = 1$.

It turns out the hyperbolic functions $(x, y) = (\cosh(t), \sinh(t))$ for $t \in (-\infty, \infty)$ trace out a branch of the standard hyperbola defined by $x^2 - y^2 = 1$, because of the basic hyperbolic identity $\cosh^2(t) - \sinh^2(t) = 1$.



In fact, the shaded sector with corners $(0,0)$, $(1,0)$, $(\cosh(t), \sinh(t))$ has area $\frac{1}{2}t$; just as in the circle, the sector with corners $(0,0)$, $(1,0)$, $(\cos(t), \sin(t))$ has area $\frac{1}{2}t$.

The basic hyperbolic identity can easily be checked from the definitions:

$$\begin{aligned}\cosh^2(x) - \sinh^2(x) &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{(e^{2x}+2+e^{-2x}) - (e^{2x}-2+e^{-2x})}{4} = \frac{4}{4} = 1\end{aligned}$$

Formulas. The analogy goes much further: almost every formula involving trigonometric functions has a hyperbolic counterpart, often with changes in the $\pm$ signs.

$$\cosh^2(x) - \sinh^2(x) = 1$$
$$\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$
$$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

$$\sinh'(x) = \cosh(x) \qquad \cosh'(x) = \sinh(x)$$
$$\tanh'(x) = \operatorname{sech}^2(x) \qquad \operatorname{sech}'(x) = -\tanh(x)\operatorname{sech}(x)$$

Each of these can be easily verified from the definitions via exponentials. For example:

$$\sinh'(x) = \left(\tfrac{1}{2}(e^x - e^{-x})\right)' = \tfrac{1}{2}\left(e^x - (-e^{-x})\right) = \cosh(x).$$

$$\begin{aligned}\tanh'(x) &= \left(\frac{\sinh(x)}{\cosh(x)}\right)' = \frac{\sinh'(x)\cosh(x) - \sinh(x)\cosh'(x)}{\cosh^2(x)} \\ &= \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)} = \operatorname{sech}^2(x).\end{aligned}$$

EXAMPLE: If $\sinh(x) = -3$, find $\cosh(x)$. Solve $\cosh^2(x) - \sinh^2(x) = 1$ to get $\cosh(x) = \pm\sqrt{1+(-3)^2} = \pm\sqrt{10}$; but $\cosh(x) = \frac{1}{2}(e^x + e^{-x}) > 0$ for all x, so $\cosh(x) = \sqrt{10}$.

EXAMPLE: Find the derivative of $\ln(\cosh(x))$. Using the Chain Rule:

$$[\ln(\cosh(x))]' \;=\; \ln'(\cosh(x)) \cdot \cosh'(x) \;=\; \tfrac{1}{\cosh(x)} \cdot \sinh(x) \;=\; \tanh(x),$$

a signed analog of $[\ln(\cos(x))]' = [-\ln(\sec(t))]' = -\tan(x)$.

EXAMPLE: Find the antiderivative $\int \frac{\sinh(x)}{\cosh^2(x)}\, dx$. Substitute $u = \cosh(x)$, $du = \sinh(x)\, dx$:

$$\int \frac{\sinh(x)}{\cosh^2(x)}\, dx \;=\; \int \frac{1}{\cosh^2(x)} \sinh(x)\, dx \;=\; \int \frac{1}{u^2}\, du \;=\; -\frac{1}{u} \;=\; -\frac{1}{\cosh(x)} \;=\; -\mathrm{sech}(x)\,.$$

(For brevity, we neglect the arbitrary constant term $+C$.) Alternatively: $\int \frac{\sinh(x)}{\cosh^2(x)}\, dx = \int \tanh(x)\,\mathrm{sech}(x)\, dx \;=\; -\mathrm{sech}(x)$, directly from reversing our derivative table.

EXAMPLE: Find $\int \sinh^2(t)\, dt$. Exactly as for $\int \sin^2(\theta)\, d\theta$, use

$$\cosh(2t) \;=\; \cosh^2(t) + \sinh^2(t) \;=\; 2\sinh^2(t) + 1,$$

so that $\sinh^2(t) = \frac{1}{2}\cosh(2t) - \frac{1}{2}$, and:

$$\int \sinh^2(t)\, dt \;=\; \int (\tfrac{1}{2}\cosh(2t) - \tfrac{1}{2})\, dt \;=\; \tfrac{1}{4}\sinh(2t) - \tfrac{1}{2}t \;=\; \tfrac{1}{2}\cosh(t)\sinh(t) - \tfrac{1}{2}t.$$

EXAMPLE: Find $\int \mathrm{sech}(x)\, dx$. The tricks for $\int \sec(x)\, dx$ do not work. Instead, write in terms of exponentials, and substitute $u = e^x$, $x = \ln(u)$, $dx = \frac{1}{u}\, du$:

$$\int \mathrm{sech}(x)\, dx \;=\; \int \tfrac{2}{e^x + e^{-x}}\, dx \;=\; \int \tfrac{2e^x}{e^{2x}+1}\, dx \;=\; \int \tfrac{2u}{u^2+1}\, \tfrac{1}{u}\, du \;=\; 2\tan^{-1}(u) \;=\; 2\tan^{-1}(e^x).$$

Inverse hyperbolic functions. We can define inverse hyperbolic functions and compute their derivatives just as for trig functions in §6.6, getting several more antiderivatives. For example, setting $y = \sinh(x)$, $x = \sinh^{-1}(y)$, we have $\cosh(x) = \sqrt{y^2+1}$ by the basic hyperbolic identity. We take $\frac{d}{dy}$ of $y = \sinh(\sinh^{-1}(y))$ to find:

$$1 \;=\; \cosh(\sinh^{-1}(y))\,(\sinh^{-1})'(y)$$

$$(\sinh^{-1})'(y) \;=\; \frac{1}{\cosh(\sinh^{-1}(y))} \;=\; \frac{1}{\cosh(x)} \;=\; \frac{1}{\sqrt{y^2+1}}.$$

Therefore:

$$\int \frac{1}{\sqrt{1+y^2}}\, dy \;=\; \sinh^{-1}(y) + C.$$

We can get a more elementary form for $x = \sinh^{-1}(y)$ by solving the equation:

$$y \;=\; \sinh(x) \;=\; \tfrac{1}{2}(e^x - e^{-x}) \;=\; \tfrac{1}{2}(e^x - \tfrac{1}{e^x}) \;=\; \tfrac{(e^x)^2 - 1}{2e^x}.$$

That is, $(e^x)^2 - 2y(e^x) - 1 = 0$, so the Quadratic Formula gives $e^x = \frac{1}{2}(2y \pm \sqrt{4y^2+4})$,

$$\sinh^{-1}(y) = x = \ln(y \pm \sqrt{y^2+1}).$$

Here $\pm$ must be $+$ since the input of logarithm must be positive.

We summarize a number of similar formulas, omitting $+C$.

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-1}}\,dx &= \cosh^{-1}(x) &= \ln(x+\sqrt{x^2-1}) \\
\int \frac{1}{\sqrt{x^2+1}}\,dx &= \sinh^{-1}(x) &= \ln(x+\sqrt{x^2+1}) \\
\int \frac{-1}{x\sqrt{1-x^2}}\,dx &= \operatorname{sech}^{-1}(x) &= -\ln(x)+\ln(1+\sqrt{1-x^2}) \\
\int \frac{-1}{x\sqrt{1+x^2}}\,dx &= \operatorname{csch}^{-1}(x) &= -\ln(x)+\ln(1+\sqrt{1+x^2}) \\
\int \frac{1}{1-x^2}\,dx &= \tanh^{-1}(x) &= \tfrac{1}{2}\ln(1+x)-\tfrac{1}{2}\ln(1-x)
\end{aligned}$$

We sometimes denote $\sin^{-1}$ as arcsin, and we may denote $\sinh^{-1}$ as arsinh or arsh, etc.

EXAMPLE: Integrate $\int\frac{1}{\sqrt{x^2+x}}\,dx$. We want to manipulate this into one of the above forms. Completing the square, we have

$$x^2+x \;=\; x^2+2(\tfrac{1}{2})x+(\tfrac{1}{2})^2-(\tfrac{1}{2})^2 \;=\; (x+\tfrac{1}{2})^2-\tfrac{1}{4} \;=\; \tfrac{1}{4}((2x+1)^2-1),$$

so the substitution $u=2x{+}1$, $du=2\,dx$ gives:

$$\int\frac{1}{\sqrt{x^2+x}}\,dx \;=\; \int\frac{1}{\sqrt{(2x{+}1)^2-1}}\,2\,dx \;=\; \int\frac{1}{\sqrt{u^2-1}}\,du$$

$$=\; \cosh^{-1}(u) \;=\; \cosh^{-1}(2x{+}1) \;=\; \ln\Big(2x+1+2\sqrt{x^2{+}x}\Big)\;.$$