Inverse Trigonometric Functions

Inverses and domains. Consider a hot-air balloon 20 feet in the air, tethered by a rope stretching 50 feet diagonally to the ground. What is the rope’s angle of elevation?

Because sine = opposite/hypotenuse, the angle of elevation \( \theta \) has \( \sin(\theta) = \frac{20}{50} = \frac{2}{5} \). To find \( \theta \), we need the inverse function: \( \theta = \sin^{-1}\left(\frac{2}{5}\right) \approx 0.41 \) rad \( \approx 23.6^\circ \), using the \text{inv} \sin or \text{arcsin} function on a calculator. However, the equation \( \sin(\theta) = \frac{2}{5} \) has infinitely many solutions:

If the initial solution is \( \theta_0 \), there is another solution at \( \theta_1 = \pi - \theta_0 \), and in general at \( \theta_0 + 2n\pi, \theta_1 + 2n\pi \) for any integer \( n \). In our problem, we clearly want an acute angle, so we restrict \( 0 \leq \theta \leq \frac{\pi}{2} \), making \( \theta = \theta_0 \) the unique acceptable solution.

A bit more generally, we restrict \( \sin(x) \) to the domain \( -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \) marked below, to make it a one-to-one function (so different inputs go to different outputs, and the graph satisfies the horizontal line test). We get a pair of inverse functions:

\[
\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1], \quad \sin^{-1} : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].
\]

See the end of this section for graphs of inverse functions with standard domains.

An alternative notation is \( \sin^{-1}(y) = \arcsin(y) \), meaning the arc (angle) whose sine is \( y \). Similarly \( \tan^{-1}(y) = \arctan(y) \), etc. Watch out for an unfortunate ambiguity: \( \sin^{-1}(x) \) could mean either \( \arcsin(x) \), the inverse under composition of functions; or \( \frac{1}{\sin(x)} \), the inverse under multiplication of functions. We will always write:

\[
\sin^{-1}(x) = \arcsin(x), \quad \sin(x)^{-1} = \frac{1}{\sin(x)} = \csc(x).
\]

Inverse functions and triangles. The Pythagorean relations between trig functions lead to relations among their inverses. Given \( \theta = \sin^{-1}(y) \), i.e. \( \sin(\theta) = y \), we set up the triangle at left below so that \( \sin(\theta) = \text{opposite/hypotenuse} = \frac{y}{1} \). The adjacent side \( x \) satisfies \( x^2 + y^2 = 1 \), so \( x = \sqrt{1 - y^2} \), and we can compute:

\[
\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{1 - y^2}}{1},
\]

that is, \( \cos(\sin^{-1}(y)) = \cos(\theta) = \sqrt{1 - y^2} \), and similarly for \( \tan(\theta) = \tan(\sin^{-1}(y)) \), etc.

\[\text{Recall that the radian angle } \theta \text{ is defined as the length of an arc on the unit circle: the full circle has circumference } 2\pi, \text{ hence } 2\pi \text{ radians.}\]
In the right picture, we have $\theta = \tan^{-1}(y)$ since $\tan(\theta) = \text{opposite/adjacent} = y/1$, and we compute $\sin(\tan^{-1}(y)) = \sin(\theta) = \frac{y}{\sqrt{1+y^2}}$, etc.

$$\theta = \sin^{-1}(y) \quad \theta = \tan^{-1}(y)$$

$$\sin(\theta) = y \quad \sin(\theta) = \frac{y}{\sqrt{1+y^2}}$$

$$\cos(\theta) = \frac{1}{\sqrt{1-y^2}} \quad \cos(\theta) = \frac{1}{\sqrt{1+y^2}}$$

$$\tan(\theta) = \frac{y}{\sqrt{1-y^2}} \quad \tan(\theta) = y$$

**Derivatives of inverse functions.** Recall the Inverse Derivative Formula from §6.1: if $y = f(\theta)$ and $\theta = f^{-1}(y)$, then:

$$(f^{-1})'(y) = \frac{1}{f'(\theta)} = \frac{1}{f'(f^{-1}(y))}.$$  

Taking $f(\theta) = \sin(\theta)$ and $\theta = \sin^{-1}(y)$, we get:

$$(\sin^{-1})'(y) = \frac{1}{\sin'(\theta)} = \frac{1}{\cos(\theta)} = \frac{1}{\sqrt{1-y^2}}.$$  

Similarly, we conclude:

$$(\sin^{-1})'(y) = \frac{1}{\sqrt{1-y^2}} \quad (\cos^{-1})'(y) = -\frac{1}{\sqrt{1-y^2}}$$

$$(\tan^{-1})'(y) = \frac{1}{1+y^2} \quad (\sec^{-1})'(y) = \frac{1}{y\sqrt{y^2-1}}.$$  

The inverse secant is widely deprecated. Instead of $\int\frac{1}{y\sqrt{y^2-1}} dy = \sec^{-1}(y)$, many prefer:

$$\left(\tan^{-1}\sqrt{y^2-1}\right)' = \frac{1}{y\sqrt{y^2-1}} \implies \int\frac{1}{y\sqrt{y^2-1}} dy = \tan^{-1}\sqrt{y^2-1}.$$
Inverse functions and integrals. The above derivative formulas can be reversed to give antiderivatives (indefinite integrals). That is, \( \int \frac{1}{\sqrt{1-y^2}} \, dy = \sin^{-1}(y) + C, \) etc.

**Example:** Find \( \int \frac{1}{\sqrt{2-x^2}} \, dx. \) The trick is to rewrite the integrand in the form of one of our derivatives, whichever is closest, in this case \( \frac{1}{\sqrt{1-y^2}}. \)

\[
\int \frac{1}{\sqrt{2-x^2}} \, dx = \int \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}} \, dx = \int \frac{1}{\sqrt{2}} \, dy = \frac{\sqrt{2}}{2} \, dx 
\]

\[
= \int \frac{1}{\sqrt{1-y^2}} \, dy = \sin^{-1}(y) + C = \sin^{-1}\left(\frac{x}{\sqrt{2}}\right) + C.
\]

**Example:** Find \( \int \frac{1}{\sqrt{1+x-x^2}} \, dx. \) Again, we want to force the integrand into the form \( \frac{1}{\sqrt{1-y^2}}. \) Since we have a three-term quadratic, we complete the square,\(^1\) writing:

\[
1 + x - x^2 = \frac{5}{4} - (x-\frac{1}{2})^2 = \frac{5}{4}\left(1 - \left(\frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}\right)^2\right).
\]

Thus, we take \( y = \frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}, \) \( dy = \frac{2}{\sqrt{5}} \, dx, \) so that:

\[
\int \frac{1}{\sqrt{1+x-x^2}} \, dx = \int \frac{1}{\sqrt{5}} \, dx = \int \frac{1}{\sqrt{1 - \left(\frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}\right)^2}} \, dx = \int \frac{1}{\sqrt{1 - \left(\frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}\right)^2}} \, dx 
\]

\[
= \int \frac{1}{\sqrt{1-y^2}} \, dy = \sin^{-1}(y) + C = \sin^{-1}\left(\frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}\right) + C.
\]

An impressive integral!

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\( ^1 \)That is, rewrite: \( x^2 + bx + c = x^2 + 2(\frac{b}{2})x + (\frac{b}{2})^2 - (\frac{b}{2})^2 + c = (x + \frac{b}{2})^2 + \frac{b^2-4c}{4}. \) This is the computation which leads to the Quadratic Formula.
The strange standard domain for \( \sec(\theta) \) is \( \theta \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2}) \), which is chosen to make the signs work out in \( (\sec^{-1})'(y) = \frac{1}{y\sqrt{y^2-1}} \). If we had instead chosen \( \sec^{-1}(y) = \cos^{-1}\left(\frac{1}{y}\right) \), we would get \( (\sec^{-1})'(y) = \frac{1}{|y|\sqrt{y^2-1}} \).