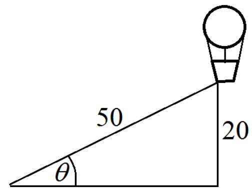
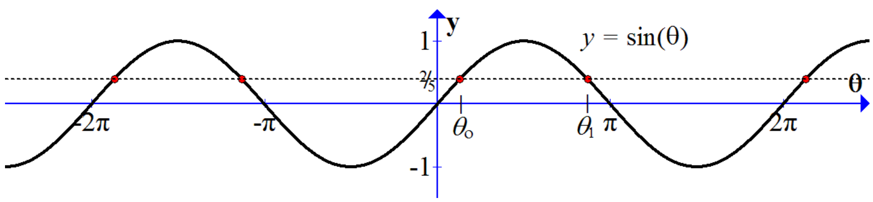
Math 133 # Inverse Trigonometric Functions **Stewart §6.6**

Inverses and domains. Consider a hot-air balloon 20 feet in the air, tethered by a rope stretching 50 feet diagonally to the ground. What is the rope's angle of elevation?



Because sine = opposite/hypotenuse, the angle of elevation θ has $\sin(\theta) = \frac{20}{50} = \frac{2}{5}$. To find θ, we need the inverse function: $\theta = \sin^{-1}(\frac{2}{5}) \approx 0.41$ rad $\approx 23.6°$, using the `inv sin` or `arcsin` function on a calculator. However, the equation $\sin(\theta) = \frac{2}{5}$ has infinitely many solutions:



If the initial solution is θ_0, there is another solution at $\theta_1 = \pi - \theta_0$, and in general at $\theta_0 + 2n\pi$, $\theta_1 + 2n\pi$ for any integer n. In our problem, we clearly want an acute angle, so we restrict $0 \le \theta \le \frac{\pi}{2}$, making $\theta = \theta_0$ the unique acceptable solution.

A bit more generally, we restrict $\sin(x)$ to the domain $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ to make it a one-to-one function (so different inputs go to different outputs, and the graph satisfies the horizontal line test). We get a pair of inverse functions:

$$\sin : \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right] \longrightarrow [-1, 1], \qquad \sin^{-1} : [-1, 1] \longrightarrow \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right].$$

See the end of this section for graphs of inverse functions with standard domains.

An alternative notation is $\sin^{-1}(y) = \arcsin(y)$, meaning the arc (angle) whose sine is y.* Similarly $\tan^{-1}(y) = \arctan(y)$, etc. Watch out for an unfortunate ambiguity: $\sin^{-1}(x)$ could mean either $\arcsin(x)$, the inverse under composition of functions; or $\frac{1}{\sin(x)}$, the inverse under multiplication of functions. We will always write:

$$\sin^{-1}(x) \;=\; \arcsin(x), \qquad \sin(x)^{-1} \;=\; \frac{1}{\sin(x)} \;=\; \csc(x)\,.$$

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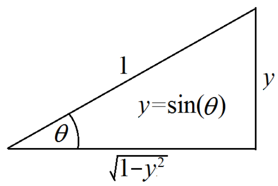
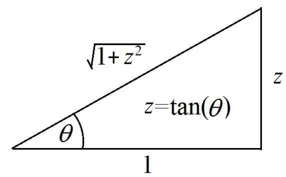
*Radian angle θ means the length of an arc on the unit circle: a full circle has circumference 2π rad.

Inverse functions and triangles. The Pythagorean relations between trig functions lead to relations among their inverses. Given $\theta = \sin^{-1}(y)$, i.e. $\sin(\theta) = y$, we set up the triangle at left below so that $\sin(\theta) =$ opposite/hypotenuse $= y/1$.





At left, the adjacent side x satisfies $x^2 + y^2 = 1$, so $x = \sqrt{1-y^2}$, and we can compute:

$$\cos(\theta) \;=\; \frac{\text{adjacent}}{\text{hypotenuse}} \;=\; \frac{\sqrt{1-y^2}}{1},$$

so $\cos(\sin^{-1}(y)) = \cos(\theta) = \sqrt{1-y^2}$; similarly $\tan(\theta) = \tan(\sin^{-1}(y)) = \frac{y}{\sqrt{1-y^2}}$, etc.

In the picture at right, we have $\theta = \tan^{-1}(z)$ since $\tan(\theta) =$ opposite/adjacent $= z/1$, and we compute $\cos(\tan^{-1}(z)) = \cos(\theta) = \frac{1}{\sqrt{1+z^2}}$, etc.

$$\begin{aligned} &\theta = \sin^{-1}(y) & \qquad & \theta = \tan^{-1}(z) \\ \sin(\theta) &= y & \tan(\theta) &= z \\ \cos(\theta) &= \sqrt{1-y^2} & \cos(\theta) &= \tfrac{1}{\sqrt{1+z^2}} \\ \tan(\theta) &= \tfrac{y}{\sqrt{1-y^2}} & \sec(\theta) &= \sqrt{1+z^2} \end{aligned}$$

Derivatives of Inverses. As in §6.1, we differentiate the defining formula $y = \sin(\sin^{-1}(y))$:

$$1 \;=\; [\sin(\sin^{-1}(y))]' \;=\; \cos(\sin^{-1}(y))\,(\sin^{-1})'(y),$$

$$(\sin^{-1})'(y) \;=\; \frac{1}{\cos(\sin^{-1}(y))} \;=\; \frac{1}{\cos(\theta)} \;=\; \frac{1}{\sqrt{1-y^2}},$$

where $\theta = \sin^{-1}(y)$, $\sin(\theta) = y$, $\cos(\theta) = \sqrt{1-y^2}$. Similarly, we conclude:

$$\begin{aligned} (\sin^{-1})'(y) &= \frac{1}{\sqrt{1-y^2}} & \qquad (\cos^{-1})'(y) &= -\frac{1}{\sqrt{1-y^2}} \\ (\tan^{-1})'(y) &= \frac{1}{1+y^2} & (\sec^{-1})'(y) &= \frac{1}{y\sqrt{y^2-1}}. \end{aligned}$$

Inverse functions and integrals. The above derivative formulas can be reversed to give antiderivatives (indefinite integrals). That is, $\int \frac{1}{\sqrt{1-y^2}}\,dy = \sin^{-1}(y) + C$, etc.

EXAMPLE: Find $\int \frac{1}{\sqrt{2-x^2}}\,dx$. The trick is to rewrite the integrand in the form of one of our derivatives, whichever is closest, in this case $\frac{1}{\sqrt{1-y^2}}$.

$$\int \frac{1}{\sqrt{2-x^2}}\,dx \;=\; \int \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1-\frac{x^2}{2}}}\,dx \;=\; \int \frac{1}{\sqrt{1-(\frac{x}{\sqrt{2}})^2}}\,\frac{1}{\sqrt{2}}\,dx \qquad \left[\begin{array}{r} y = \frac{x}{\sqrt{2}} \\ dy = \frac{1}{\sqrt{2}}\,dx \end{array}\right]$$

$$= \int \frac{1}{\sqrt{1-y^2}}\,dy \;=\; \sin^{-1}(y) + C \;=\; \sin^{-1}(\tfrac{x}{\sqrt{2}}) + C\,.$$

EXAMPLE: Find $\int \frac{1}{\sqrt{1+x-x^2}}\,dx$. Again, we want to force the integrand into the form $\frac{1}{\sqrt{1-y^2}}$. Since we have a three-term quadratic, we complete the square:†

$$x^2 - x - 1 \;=\; x^2 - 2(\tfrac{1}{2})x + (\tfrac{1}{2})^2 - (\tfrac{1}{2})^2 - 1 \;=\; (x-\tfrac{1}{2})^2 - \tfrac{5}{4},$$

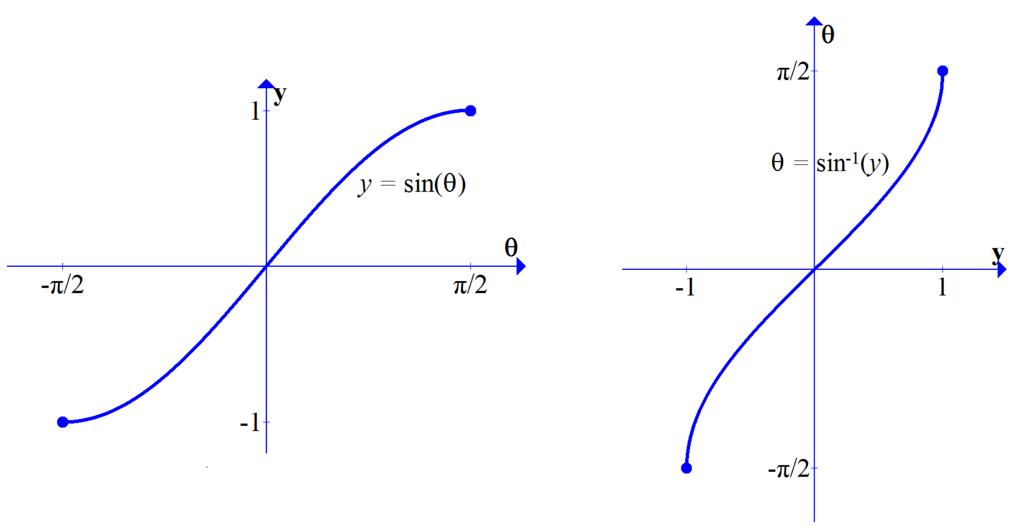
$$1 + x - x^2 \;=\; \tfrac{5}{4} - (x-\tfrac{1}{2})^2 \;=\; \tfrac{5}{4}\Big(1 - \tfrac{4}{5}\,(x-\tfrac{1}{2})^2\Big) \;=\; \tfrac{5}{4}\Big(1 - \big(\tfrac{2}{\sqrt{5}}x - \tfrac{1}{\sqrt{5}}\big)^2\Big)\,.$$

Thus we take $y = \frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}$, $dy = \frac{2}{\sqrt{5}}\,dx$, and obtain the impressive integral:

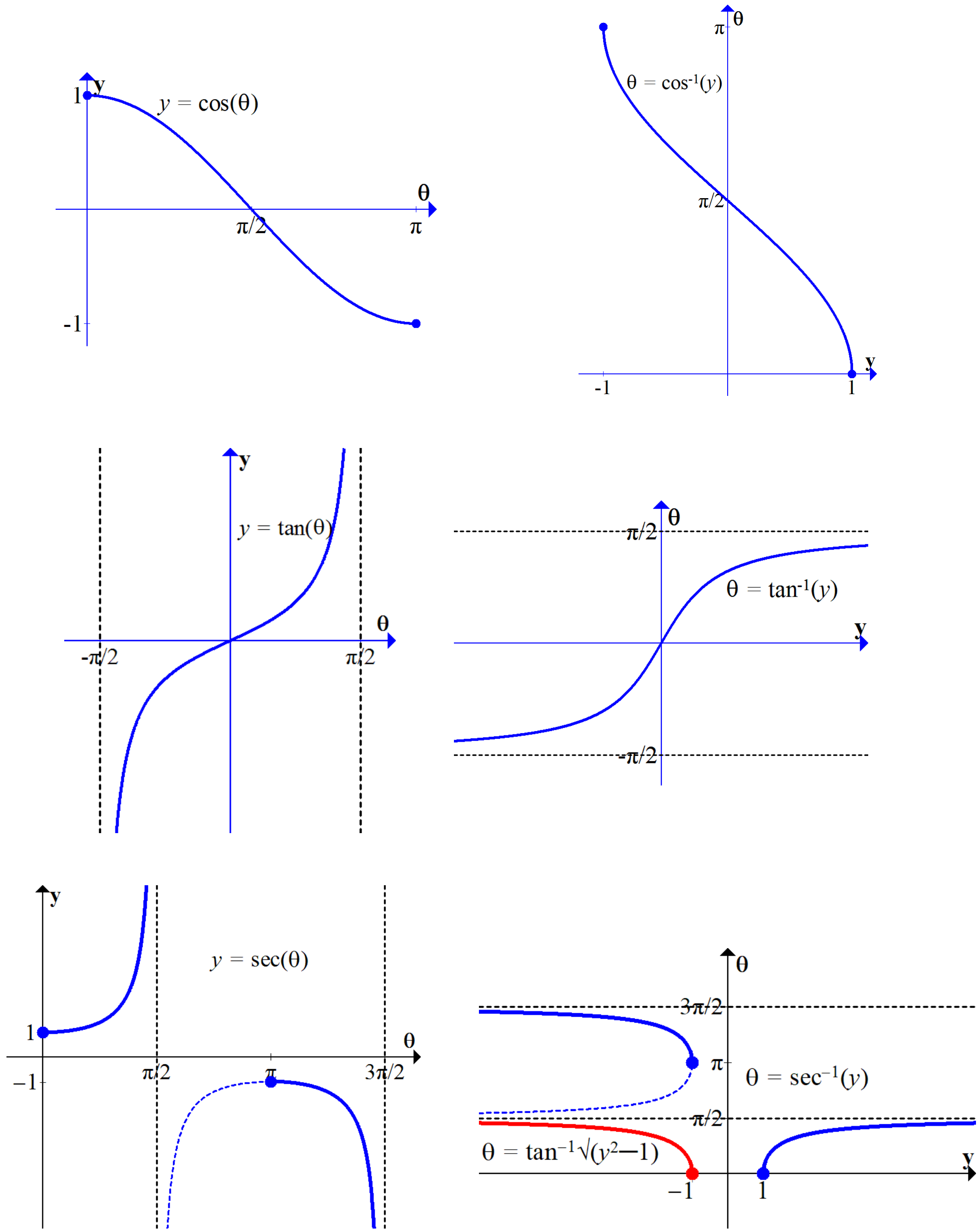
$$\int \frac{1}{\sqrt{1+x-x^2}}\,dx \;=\; \int \frac{1}{\sqrt{\frac{5}{4}\Big(1-\big(\frac{2}{\sqrt{5}}x-\frac{1}{\sqrt{5}}\big)^2\Big)}}\,dx \;=\; \int \frac{1}{\sqrt{1-\big(\frac{2}{\sqrt{5}}x-\frac{1}{\sqrt{5}}\big)^2}}\,\frac{2}{\sqrt{5}}\,dx$$

$$= \int \frac{1}{\sqrt{1-y^2}}\,dy \;=\; \sin^{-1}(y) + C \;=\; \sin^{-1}\Big(\tfrac{2}{\sqrt{5}}x - \tfrac{1}{\sqrt{5}}\Big) + C\,.$$

Graphs of inverse functions



† $x^2 + bx + c \;=\; x^2 + 2(\frac{b}{2})x + (\frac{b}{2})^2 - (\frac{b}{2})^2 + c \;=\; (x + \frac{b}{2})^2 + \frac{b^2-4c}{4}$, leading to the Quadratic Formula.



The strange standard domain for $\sec(\theta)$ is $\theta \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$, chosen to make signs work out in $(\sec^{-1})'(y) = \frac{1}{y\sqrt{y^2-1}}$. If we took $\theta \in [0, \pi]$, we would get $(\sec^{-1})'(y) = \frac{1}{|y|\sqrt{y^2-1}}$. To avoid this headache, we usually write $\int \frac{1}{y\sqrt{y^2-1}}\, dy = \tan^{-1}\sqrt{y^2-1}$, not $\sec^{-1}(y)$. Indeed, for $y \geq 1$ these are the same, $\tan^{-1}\sqrt{y^2-1} = \sec^{-1}(y)$; but for $y \leq -1$ they differ by $+\pi$ (red curve). The function $\tan^{-1}\sqrt{y^2-1}$ is an even function, not an inverse, but it is unambiguously defined and has the correct derivative: $(\tan^{-1}\sqrt{y^2-1})' = \frac{1}{y\sqrt{y^2-1}}$.