Derivative of general exp. To compute with functions of arbitrary base, we will repeatedly apply:

\textit{Natural Base Principle:} To deal with general exponentials and logarithms in calculus, write them in terms of the natural base \( e \) functions \( e^x \) and \( \ln(x) \), which have \( (e^x)' = e^x \) and \( \ln'(x) = \frac{1}{x} \).

For example, we have \( a = e^{\ln(a)} \), so:

\[(a^x)' = (e^{\ln(a) \cdot x})' = \exp'(\ln(a) \cdot x) \cdot (\ln(a) \cdot x)' = e^{\ln(a) \cdot x} \cdot (\ln(a) \cdot x)' = ln(a) \cdot a^x.\]

Note that one factor is just our original function \( a^x \), because differentiating the outside function \( e^x \) has no effect. In the other factor, \( \ln(a) \) is a (complicated) constant, so \( (\ln(a) \cdot x)' = \ln(a) \).

Derivative of general log. Since \( f(x) = a^x = e^{\ln(a) \cdot x} \), we can find the inverse function \( f^{-1}(y) = \log_a(y) \) by solving \( y = e^{\ln(a) \cdot x} \) to get: \( \ln(y) = \ln(a) \cdot x \), and \( x = \frac{\ln(y)}{\ln(a)} \). That is, \( f^{-1}(y) = \log_a(y) = \frac{\ln(y)}{\ln(a)} \). Switching the input variable to \( x \), we get the \textit{logarithm base change formula}:

\[\log_a(x) = \frac{\ln(x)}{\ln(a)} .\]

Hence:

\[\log'_a(x) = \left(\frac{\ln(x)}{\ln(a)}\right)' = \frac{1}{\ln(a)} \ln'(x) = \frac{1}{\ln(a)} x.\]

Problems.

\textbf{EXAMPLE:} Differentiate \( f(x) = 6^{x+\cos(x)} \). It is \textit{not} helpful to factor: \( f(x) = 6^x 6^{\cos(x)} \).

Instead, we have \( 6 = e^{\ln(6)} \), so:

\[f'(x) = (e^{\ln(6) \cdot (x+\cos(x))})' = \exp'(\ln(6) \cdot (x+\cos(x))) \cdot \ln(6)(x+\cos(x))' = e^{\ln(6) \cdot (x+\cos(x))} \cdot \ln(6)(1-\sin(x)) = 6^{x+\cos(x)} \ln(6)(1-\sin(x)).\]

Notice that the original function is again a factor of the derivative, because the derivative of the outside exp is itself.

\textbf{EXAMPLE:} Differentiate \( f(x) = x^x \). Since \( x = e^{\ln(x)} \), we have:

\[f'(x) = (e^{\ln(x) \cdot x})' = \exp'(\ln(x) \cdot x) \cdot (\ln(x) \cdot x)' = \exp'(\ln(x) \cdot x) \cdot (\ln'(x) \cdot x + \ln(x) x') = x^x (1 + \ln(x)).\]

Once again, the original function is a factor of the derivative.
Another approach is the logarithmic derivative, based on the formula:

\[
\frac{\ln'(f(x)) f'(x)}{f(x)} = f'(x) = f(x) (\ln(f(x)))'.
\]

For our function, \(\ln(f(x)) = \ln(x^2) = x \ln(x)\), and we quickly get the previous answer:

\[
f'(x) = f(x) (\ln(f(x)))' = x^2(x \ln(x))' = x^2(1 + \ln(x)).
\]

**EXAMPLE:** Find the indefinite integral\(^*\) \(\int x^2 x^2 \, dx\).

We write in terms of natural functions, and do the substitution \(u = \ln(6) x^2\):

\[
\int x^2 x^2 \, dx = \int x e^{\ln(6) x^2} \, dx = \frac{1}{2 \ln(6)} \int e^{\ln(6) x^2} \ln(6) 2x \, dx
\]

\[
= \frac{1}{2 \ln(6)} \int e^u \, du = \frac{e^u}{2 \ln(6)} = \frac{e^{\ln(6) x^2}}{2 \ln(6)} = \frac{6x^2}{2 \ln(6)}
\]

\(\int f(x) \, dx\), with no limits of integration, is simply a shorthand for the general antiderivative, and is called the *indefinite integral*. Indeed, if we find the indefinite integral \(\int f(x) \, dx = F(x) + C\), where \(F'(x) = f(x)\), then we can evaluate the definite integral: \(\int_a^b f(x) \, dx = [F(x)]_a^b\).