Basic Properties. Here is pretty much all you need to know about the \( \exp(x) \) and \( \ln(x) \) functions.

- \( \exp(x) = e^x \) \( \ln(x) = \log_e(x) \)
- \( e^\ln(x) = x \) \( \ln(e^x) = x \)
- \( e^0 = 1 \) \( e^1 = e \approx 2.71 \) \( \ln(1) = 0 \) \( \ln(e) = 1 \)
- \( e^{x_1} e^{x_2} = e^{x_1 + x_2} \) \( (e^x)^p = e^{px} \)
- \( \ln(x_1 x_2) = \ln(x_1) + \ln(x_2) \) \( \ln(x^p) = p \ln(x) \)
- \( (e^x)' = e^x \) \( \int e^x \, dx = e^x + C \) \( \ln'(x) = \frac{1}{x} \) \( \int \frac{1}{x} \, dx = \ln|x| + C \)

We give some tricky examples, applying the basic facts and the Chain Rule.

**EXAMPLE:** Solve for \( y \) in the equation: \( \ln(y e^x) + 1 = 2x + \ln(y^2) \).
Strategy: Expand into a sum, move \( y \)’s to the left, all else to the right.

\[
\ln(y) + \ln(e^x) + 1 = 2x + 2 \ln(y) \\
\ln(y) - 2 \ln(y) = 2x - x - 1 \\
\ln(y) = 1 - x \\
y = e^{1-x}.
\]

**EXAMPLE:** Differentiate \( f(x) = \sin(e^{\tan(x)}) \).
Strategy: Apply the Chain Rule with outer = \( \sin(x) \), inner = \( e^{\tan(x)} \).

\[
\left(\sin(e^{\tan(x)})\right)' = \sin'(e^{\tan(x)}) \cdot (e^{\tan(x)})' \\
= \cos(e^{\tan(x)}) \cdot \exp'(\tan(x)) \cdot \tan'(x) \\
= \cos(e^{\tan(x)}) \cdot e^{\tan(x)} \cdot \sec^2(x)
\]

**EXAMPLE:** Differentiate \( f(x) = \int_2^{e^x} \ln(t \sin(t)) \, dt \).
Strategy: Apply the Chain Rule with outer function \( g(x) = \int_2^{e^x} \ln(t \sin(t)) \, dt \). The First Fundamental Theorem (§4.3) says* that \( g'(x) = \ln(x \sin(x)) \). We are given a composition of functions \( f(x) = g(e^x) \), so the Chain Rule applies:

\[
f'(x) = g'(e^x) \cdot (e^x)' = \ln(e^x \sin(e^x)) \cdot e^x = xe^x + \ln(\sin(e^x)) e^x.
\]

*That is, in the plane with \( t \) and \( y \) axes, \( g(x) \) is the area between the curve \( y = \ln(t \sin(t)) \) and the interval \( t \in [1, x] \). The rate of change of this area function, \( g'(x) \), equals the level of the curve at \( t = x \), the moving end of the interval: \( \ln(x \sin(x)) \).

Algebraically, the derivative of the integral of a function gives back the original function.
EXAMPLE: Find $\frac{dy}{dx}$ by implicit differentiation, for $(x, y)$ satisfying:

$$e^y = \cos(x+y).$$

Specifically, find $\frac{dy}{dx}$ at the point $(x, y) = (0, 0)$.

Strategy: The equation defines some unknown curve containing the point $(x, y) = (0, 0)$, since $e^0 = \cos(0+0)$. We want the slope of the tangent line at that point. Assuming $y = y(x)$ is some function which satisfies the equation, we apply the Chain Rule to both sides, and solve for $y' = \frac{dy}{dx}$.

$$\left(e^{y(x)}\right)' = \cos(x+y(x))'$$

$$\exp'(y(x)) \cdot y'(x) = \cos'(x+y(x)) \cdot (x+y(x))'$$

$$e^y \cdot y' = -\sin(x+y) \cdot (1+y')$$

$$(e^y + \sin(x+y))y' = -\sin(x+y)$$

$$y' = \frac{-\sin(x+y)}{e^y + \sin(x+y)}.$$  

Substituting $(x, y) = (0, 0)$ gives $y' = \frac{dy}{dx} \bigg|_{x=0} = -\frac{\sin(0+0)}{e^0 + \sin(0+0)} = 0$. That is, the unknown curve has a horizontal tangent at the origin.

EXAMPLE: Find the derivative of $f(x) = a^x$ for any base $a > 0$.

Strategy: write $a^x$ in terms of the natural exponential, whose derivative is known. Specifically, solving $a = e^p$ by $p = \ln(a)$, we get $a = e^{\ln(a)}$, and $a^x = (e^{\ln(a)})^x = e^{\ln(a) \cdot x}$. Applying the Chain Rule:

$$\left(a^x\right)' = \left(e^{\ln(a) \cdot x}\right)' = \exp' \left(\ln(a) \cdot x\right) \cdot \left(\ln(a) \cdot x\right)'$$

$$= e^{\ln(a) \cdot x} \cdot \ln(a) = \ln(a) \cdot a^x.$$  

Note that $\ln(a)$ is a constant, so $(\ln(a) \cdot x)' = \ln(a).$  

\[1\text{If we tried to apply the Product and Chain Rules, we would get:}

$$\left(\ln(a) \cdot x\right)' = (\ln(a))' \cdot x + \ln(a) \cdot (x)' = \ln'(a) \cdot x + \ln(a) \cdot 1 = \frac{1}{e^x} \cdot x + \ln(a) = \ln(a).$$