Math 133 **Natural Exp and Log** **Stewart §6.3**

Natural Exp and Log

Basic Properties. Here is pretty much all you need to know about the $\exp(x)$ and $\ln(x)$ functions.

- $\exp(x) = e^x \qquad \ln(x) = \log_e(x)$
- $e^{\ln(x)} = x \qquad \ln(e^x) = x$
- $e^0 = 1 \qquad e^1 = e \approx 2.71 \qquad \ln(1) = 0 \qquad \ln(e) = 1$
- $e^{x_1} e^{x_2} = e^{x_1+x_2} \qquad (e^x)^p = e^{px}$
- $\ln(x_1 x_2) = \ln(x_1) + \ln(x_2) \qquad \ln(x^p) = p\ln(x)$
- $(e^x)' = e^x \qquad \int e^x\,dx = e^x + C \qquad \ln'(x) = \frac{1}{x} \qquad \int \frac{1}{x}\,dx = \ln|x| + C$

We give some tricky examples, applying the basic facts and the Chain Rule.

EXAMPLE: Solve for y in the equation: $\ln(ye^x) + 1 = 2x + \ln(y^2)$.
Strategy: Expand into a sum, move y's to the left, all else to the right.

$$\ln(y) + \ln(e^x) + 1 = 2x + 2\ln(y)$$
$$\ln(y) - 2\ln(y) = 2x - x - 1$$
$$\ln(y) = 1 - x$$
$$y = e^{1-x}.$$

EXAMPLE: Differentiate $f(x) = \sin(e^{\tan(x)})$.
Strategy: Apply the Chain Rule with outer $= \sin(x)$, inner $= e^{\tan(x)}$.

$$\begin{aligned}\left(\sin(e^{\tan(x)})\right)' &= \sin'(e^{\tan(x)}) \cdot (e^{\tan(x)})' \\ &= \cos(e^{\tan(x)}) \cdot \exp'(\tan(x)) \cdot \tan'(x) \\ &= \cos(e^{\tan(x)}) \cdot e^{\tan(x)} \cdot \sec^2(x)\end{aligned}$$

EXAMPLE: Differentiate $f(x) = \displaystyle\int_2^{e^x} \ln(t\sin(t))\,dt$.
Strategy: Apply the Chain Rule with outer function $g(x) = \int_2^x \ln(t\sin(t))\,dt$. The First Fundamental Theorem (§4.3) says* that $g'(x) = \ln(x\sin(x))$. We are given a composition of functions $f(x) = g(e^x)$, so the Chain Rule applies:

$$f'(x) \;=\; g'(e^x)\cdot(e^x)' \;=\; \ln(e^x\sin(e^x))\cdot e^x \;=\; xe^x + \ln(\sin(e^x))\,e^x.$$

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*That is, in the plane with t and y axes, $g(x)$ is the area between the curve $y = \ln(t\sin(t))$ and the interval $t \in [1, x]$. The rate of change of this area function, $g'(x)$, equals the level of the curve at $t = x$, the moving end of the interval: $\ln(x\sin(x))$.

Algebraically, the derivative of the integral of a function gives back the original function.

EXAMPLE: Find $\frac{dy}{dx}$ by implicit differentiation, for (x, y) satisfying:

$$e^y \;=\; \cos(x{+}y).$$

Specifically, find $\frac{dy}{dx}$ at the point $(x, y) = (0, 0)$.
Strategy: The equation defines some unknown curve containing the point $(x, y) = (0, 0)$, since $e^0 = \cos(0{+}0)$. We want the slope of the tangent line at that point. Assuming $y = y(x)$ is some function which satisfies the equation, we apply the Chain Rule to both sides, and solve for $y' = \frac{dy}{dx}$.

$$\begin{aligned}
(e^{y(x)})' \;&=\; \cos(x + y(x))' \\
\exp'(y(x)) \cdot y'(x) \;&=\; \cos'(x + y(x)) \cdot (x + y(x))' \\
e^y \cdot y' \;&=\; -\sin(x{+}y) \cdot (1{+}y') \\
(e^y + \sin(x{+}y))\, y' \;&=\; -\sin(x{+}y) \\
y' \;&=\; -\frac{\sin(x{+}y)}{e^y + \sin(x{+}y)}\,.
\end{aligned}$$

Substituting $(x, y) = (0, 0)$ gives $y' = \left.\frac{dy}{dx}\right|_{x=0} = -\frac{\sin(0+0)}{e^0+\sin(0+0)} = 0$. That is, the unknown curve has a horizontal tangent at the origin.

EXAMPLE: Find the derivative of $f(x) = a^x$ for any base $a > 0$.
Strategy: write a^x in terms of the natural exponential, whose derivative is known. Specifically, solving $a = e^p$ by $p = \ln(a)$, we get $a = e^{\ln(a)}$, and $a^x = (e^{\ln(a)})^x = e^{\ln(a)\,x}$. Applying the Chain Rule:

$$\begin{aligned}
(a^x)' \;&=\; (e^{\ln(a)\,x})' \;=\; \exp'(\ln(a)\,x) \cdot (\ln(a)\,x)' \\
&=\; e^{\ln(a)\,x} \cdot \ln(a) \;=\; \ln(a)\,a^x\,.
\end{aligned}$$

Note that $\ln(a)$ is a constant, so $(\ln(a)x)' = \ln(a)$.[†]

[†]If we tried to apply the Product and Chain Rules, we would get:

$$(\ln(a)\,x)' = (\ln(a))' \cdot x + \ln(a) \cdot (x)' = \ln'(a) \cdot a' \cdot x + \ln(a) \cdot 1 = \tfrac{1}{a} \cdot 0 \cdot x + \ln(a) = \ln(a).$$