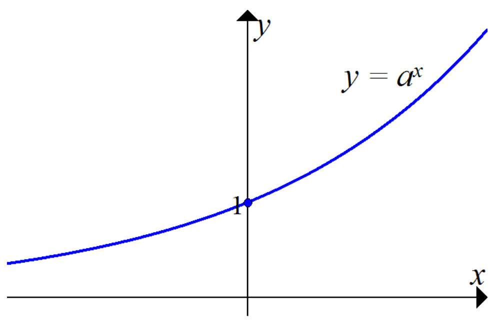
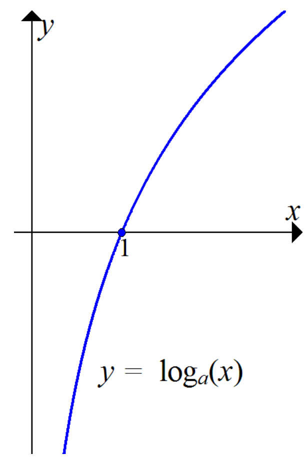
Natural Logarithms

Review of exponential and logarithm functions. We recall some facts from algebra, which we will later prove from a calculus point of view. In an expression of the form a^p, the number a is called the *base* and the power p is the *exponent*. An *exponential function*[*] is of the form $f(x) = a^x$. It is defined for rational $x = \frac{m}{n}$ by $a^{m/n} = \sqrt[n]{a \cdots a}$ (m factors), $a^{-x} = 1/a^x$, $a^0 = 1$; but this is harder to define for irrational exponents like $a^{\sqrt{2}}$. We have addition and multiplication formulas: $a^{x_1}a^{x_2} = a^{x_1+x_2}$ and $(a^x)^p = a^{px}$.

Given the exponential function $f(x) = a^x$, the *logarithm* function is the inverse $f^{-1}(x) = \log_a(x)$, as defined in §6.1.[†]





That is, the equation $y = a^x$ is by definition solved as $x = \log_a(y)$, and we have $\log_a(a^x) = x$ and $a^{\log_a(y)} = y$. Every fact about the exponential function corresponds to an inverse fact about the logarithm. Setting $y_1 = a^{x_1}$, $x_1 = \log(y_1)$; and $y_2 = a^{x_2}$, $x_2 = \log(y_2)$; the addition formula becomes:

$$\begin{aligned} a^{x_1}a^{x_2} = a^{x_1+x_2} \quad &\Longrightarrow \quad y_1y_2 = a^{\log(y_1)+\log(y_2)} \\ &\Longrightarrow \quad \log(y_1y_2) = \log(y_1) + \log(y_2). \end{aligned}$$

Setting $y = a^x$, $x = \log(y)$, the multiplication formula becomes:

$$\begin{aligned} (a^x)^p = a^{px} \quad &\Longrightarrow \quad y^p = a^{p\,\log(y)} \\ &\Longrightarrow \quad \log(y^p) = p\,\log(y). \end{aligned}$$

EXAMPLE: Expand the expression $\log\sqrt{\frac{x+1}{x-1}}$ as much as possible into a sum of simple terms. Using the addition and multiplication formulas:

$$\begin{aligned} \log\sqrt{\tfrac{x+1}{x-1}} &= \log\left((x+1)(x-1)^{-1}\right)^{1/2} \\ &= \tfrac{1}{2}\left(\log(x+1) + (-1)\log(x-1)\right) \\ &= \tfrac{1}{2}\log(x+1) - \tfrac{1}{2}\log(x-1). \end{aligned}$$

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[*] Do not confuse this with a *power function* of the form $f(x) = x^p$

[†] If the base a is understood, we write simply $\log(x)$. In science and engineering literature, if there is no base specified, we assume the base $a = 10$.

Natural exponential and logarithm. In the physical world, an exponential function $f(t) = a^t$ typically appears as the size of a population which is self-reproducing. This means the population growth rate, the number of births per unit time, is proportional to the current population size:

$$f'(t) = c\, f(t)\,.$$

It is a fact (proven below) that any exponential function $f(t) = a^t$ satisfies this equation for some constant c.[‡]

The *natural exponential function* uses the unique choice of base $a = e = 2.718\cdots$ which makes the above constant $c = 1$. That is, if we write $f(x) = \exp(x) = e^x$, then:

$$f'(x) = f(x), \quad \exp'(x) = \exp(x), \quad (e^x)' = e^x.$$

The *natural logarithm* is the inverse function of $f(x) = \exp(x)$, namely $f^{-1}(x) = \ln(x) = \log_e(x)$, so $y = e^x$ means $x = \ln(y)$. As in §6.1, to find the derivative of $\ln(x)$, we differentiate $x = \exp(\ln(x))$:

$$1 = \exp'(\ln(x)) \cdot \ln'(x) = \exp(\ln(x))\ \ln'(x) = x\ \ln'(x) \quad \Longrightarrow \quad \ln'(x) = \frac{1}{x}\,.$$

Amazingly, though the definition of $\ln(x)$ was complicated, its derivative is the extremely simple function $\frac{1}{x}$.

EXAMPLE: Find the derivative of $f(x) = \ln(x^2{+}1)$. Taking outside function $\ln(x)$ with $\ln'(x) = \frac{1}{x}$, and inside function $x^2{+}1$, the Chain Rule gives:

$$f'(x) \;=\; \ln'(x^2{+}1) \cdot (x^2{+}1)' \;=\; \frac{1}{x^2{+}1} \cdot (2x) \;=\; \frac{2x}{x^2{+}1}\,.$$

EXAMPLE: Find the derivative of $f(x) = \ln(\sin(x))$. From the Chain Rule;

$$f'(x) \;=\; \ln'(\sin(x)) \cdot \sin'(x) \;=\; \frac{1}{\sin(x)} \cdot \cos(x) \;=\; \cot(x).$$

EXAMPLE: Find the derivative of $f(x) = \dfrac{(x{+}1)^3 \sin^2(x)}{(2x{+}1)^5}$ using the shortcut of *logarithmic differentiation*, Take log of both sides, turning products into sums, then differentiate:

$$\begin{aligned}
\ln f(x) &= 3\ln(x{+}1) + 2\ln(\sin(x)) - 5\ln(2x{+}1) \\
(\ln f(x))' &= 3\ln(x{+}1)' + 2\ln(\sin(x))' - 5\ln(2x{+}1)' \\
\frac{1}{f(x)} f'(x) &= 3\,\frac{1}{x{+}1} + 2\,\frac{1}{\sin(x)}\cos(x) - 5\,\frac{1}{2x{+}1}(2) \\
f'(x) &= \frac{(x{+}1)^3 \sin^2(x)}{(2x{+}1)^5} \left(\frac{3}{x{+}1} + 2\cot(x) - \frac{10}{2x{+}1} \right).
\end{aligned}$$

Logarithms and integrals. Reversing our new basic derivative $\ln'(x) = \frac{1}{x}$, we see $\int \frac{1}{x}\, dx = \ln(x) + C$, the antiderivative of $\frac{1}{x} = x^{-1}$, a key function which we previously

[‡]Mathematical laws in science are typically stated in such *differential equations*, in which an unknown function $f(t)$ has a specified relation with its rate of change $f'(t)$, its acceleration $f''(t)$, etc. For example, Newton's law of universal gravitation is essentially $f''(t) = -c/f(t)^2$.

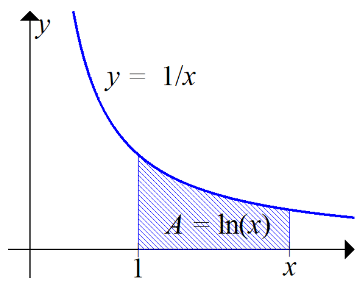
could not integrate. (The usual power function formula would give the nonsense answer $\int x^{-1}\,dx \overset{??}{=} \frac{1}{-1+1}x^{-1+1} = \frac{1}{0}x^0$.) Thus the Second Fundamental Theorem of Calculus (§4.3) tells us, for $a, b > 0$:

$$\int_a^b \frac{1}{x}\,dx \;=\; \big[\ln(x)\big]_{x=a}^{x=b} \;=\; \ln(b) - \ln(a).$$

To extend to negative x, we use:

$$\int \frac{1}{x}\,dx \;=\; \ln|x| + C \quad\Longrightarrow\quad \int_a^b \frac{1}{x}\,dx \;=\; \ln|b| - \ln|a| \quad \text{for } a, b < 0.$$

Geometrically, $\ln(x) = \int_1^x \frac{1}{t}\,dt$ is the area under $y = \frac{1}{x}$ and above the interval $[1, x]$.



Calculators and computers need an approximation algorithm to compute values of $\ln(x)$ more efficiently than just guessing solutions of $e^y = x$. The integral above allows us to approximate a natural logarithm as a Riemann sum. For example, to compute

$$\ln(2) = 0.693147\cdots,$$

split the interval $[1, 2]$ into $n = 100$ increments of size $\Delta x = \frac{2-1}{n} = 0.01$, take sample points $x_i = 1 + i\Delta x$ for $i = 1, \ldots, n$, and compute the right Riemann sum:

$$\ln(2) \;\approx\; \tfrac{1}{1.01}(0.01) + \tfrac{1}{1.02}(0.01) + \cdots + \tfrac{1}{1.99}(0.01) + \tfrac{1}{2.00}(0.01) \;\approx\; \underline{0.69}1\,,$$

which is accurate to two decimal places. The Midpoint Method, which samples midpoints $x_i = 1 + i\Delta x - \frac{\Delta x}{2}$, gives five decimal places without more computation:

$$\ln(2) \;\approx\; \tfrac{1}{1.005}(0.01) + \tfrac{1}{1.015}(0.01) + \cdots + \tfrac{1}{1.985}(0.01) + \tfrac{1}{1.995}(0.01) \;\approx\; \underline{0.69314}4\,.$$

EXAMPLE: Compute $\int_a^b \frac{\ln(x)}{x}\,dx$. Although there does not appear to be any outside or inside function, we see that $\frac{\ln(x)}{x} = \ln(x)\cdot\frac{1}{x} = \ln(x)\cdot\ln'(x)$, so we can use the substitution $u = \ln(x)$, $du = \frac{1}{x}\,dx$:

$$\int \ln(x)\,\frac{1}{x}\,dx \;=\; \int u\,du \;=\; \tfrac{1}{2}u^2 = \tfrac{1}{2}\ln^2(x)\,.$$

Thus, $\int_a^b \frac{\ln(x)}{x}\,dx = \frac{1}{2}\ln^2(b) - \frac{1}{2}\ln^2(a)$.

EXAMPLE: A tricky integral: $\int \sec(x)\,dx$. There seems to be no convenient substitution, but an amazing trick introduces $\sec^2(x) = \tan'(x)$ and $\sec(x)\tan(x) = \sec'(x)$:

$$\int \sec(x)\,dx \;=\; \int \sec(x)\,\tfrac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx \;=\; \int \tfrac{1}{\sec(x)+\tan(x)} \cdot \big(\sec^2(x) + \sec(x)\tan(x)\big)\,dx$$

This is perfect to substitute $u = \sec(x)+\tan(x)$, $du = \big(\sec^2(x) + \sec(x)\tan(x)\big)\,dx$:

$$\int \frac{1}{\sec(x)+\tan(x)}\,\big(\sec^2(x)+\sec(x)\tan(x)\big)\,dx \;=\; \int \frac{1}{u}\,du \;=\; \ln|u| \;=\; \ln|\sec(x)+\tan(x)|.$$

EXAMPLE: Time flies.[§] Suppose that the subjective length of each day is equal to the fraction of your past life it represents. Thus, when you are 1 year old, an extra day feels like $\frac{1}{365}$ of a year; but when you are 2 years old, it only feels like $\frac{1}{2\cdot 365}$ of a year; and after t years, it feels like $\frac{1}{t\cdot 365}$. Adding up the subjective lengths of all the days from $t=1$ to x years, denoting $\Delta t = \frac{1}{365}$:

$$s(x) \;\approx\; \tfrac{1}{1}\Delta t + \tfrac{1}{1+\Delta t}\Delta t + \tfrac{1}{1+2\Delta t}\Delta t + \tfrac{1}{1+3\Delta t}\Delta t + \cdots + \tfrac{1}{x}\Delta t.$$

But this is just a Riemann sum for $s(x) = \int_1^x \frac{1}{t}\,dt = \log(x)$. This means your subjective years increase logarithmically, and the time from age 2 to age $2e \approx 5.2$ feels the same as from age 10 to age $10e \approx 27$. Seems true to me!

Proofs. To formally prove the basic facts about exponentials and logarithms, we start from the one connection between these transcendental functions and an elementary function: $\ln'(x) = \frac{1}{x}$.

We now forget everything we previously stated about exponentials and logarithms, and build up our definitions from scratch, proving all properties.

Definition. For a given $x > 0$, we let: $\ln(x) = \int_1^x \frac{1}{t}\,dt$.

That is, having forgotten our previous definition of $\ln(x)$, we take the symbol $\ln(x)$ to mean the given integral, which we can compute to arbitrary accuracy with Riemann sums. Given this, the First Fundamental Theorem (§4.3) immediately proves $\ln'(x) = \frac{1}{x}$. Next:

Theorem: (a) $\ln(x_1x_2) = \ln(x_1) + \ln(x_2)$; (b) $\ln(x^p) = p\ln(x)$.

Proof. (a) For a constant $k > 0$, the derivative of $\ln(kx)$ is: $\ln(kx)' = \ln'(kx)\cdot k = \frac{1}{kx}\cdot k = \frac{1}{x} = \ln'(x)$. Since $\ln(kx)$ and $\ln(x)$ are both antiderivatives of $\frac{1}{x}$, we must have $\ln(kx) = \ln(x) + C$ for some constant (§3.9 Antiderivative Theorem). Setting $x = 1$, we get $\ln(k) = \ln(1) + C = C$, i.e. $C = \ln(k)$. Thus, $\ln(kx) = \ln(x) + \ln(k) = \ln(k) + \ln(x)$, which becomes the desired formula if we let $k = x_1$ and $x = x_2$.
(b) We use the same steps, starting from $\ln(x^p)' = \frac{1}{x^p}\,px^{p-1} = \frac{p}{x} = (p\ln(x))'$.

Definition: The function $f(x) = \ln(x)$ is one-to-one for $x > 0$, so it has an inverse function $f^{-1}(x) = \ln^{-1}(x)$. We name this inverse $\exp(x) = \ln^{-1}(x)$.

Indeed, since $\ln'(x) = \frac{1}{x} > 0$ for all $x > 0$, we know that $\ln(x)$ is increasing, i.e. $x_1 < x_2$ guarantees $\ln(x_1) < \ln(x_2)$; thus $\ln(x)$ is necessarily one-to-one. The Inverse Theorem (§6.1) immediately proves $\exp(\ln(x)) = x$ and $\ln(\exp(x)) = x$.

Theorem: $\exp'(x) = \exp(x)$

Proof. As in the Inverse Derivative Theorem (§6.1), differentiating $x = \ln(\exp(x))$ gives:

$$1 = [\ln(\exp(x))]' = \ln'(\exp(x))\cdot\exp'(x) = \frac{1}{\exp(x)}\exp'(x) \qquad \Longrightarrow \qquad \exp'(x) \;=\; \exp(x).$$

We define the number $e = \exp(1)$, i.e. the unique number such that $\int_1^e \frac{1}{t}\,dt = 1$; and the general exponential function $a^x = \exp(\ln(a)x)$, so that $e^x = \exp(x)$. The Chain Rule gives $(a^x)' = \exp(\ln(a)x)\cdot\ln(a) = \ln(a)\,a^x$. Finally, we have the exponential addition formula:

$$e^{x_1}e^{x_2} = \exp(\ln(e^{x_1}e^{x_2})) = \exp(\ln(e^{x_1}) + \ln(e^{x_2})) = e^{x_1+x_2}.$$

Also the exponential multiplication formula: $(e^x)^p = \exp(\ln(e^x)p) = \exp(xp) = e^{xp}$.

[§] Time flies like an arrow. Fruit flies like a banana.