Work

Energy and work. The concept of *energy* in physics unifies a variety of everyday concepts which can all be converted into each other by doing *work*.

- Kinetic energy is in the motion of a mass, and can do work by pushing.
- Potential energy lies in the position of an object pulled by a force field: water behind a dam can do work as gravity pulls it down.
- Chemical energy, in the ions of a battery, can work a machine; in the molecular bonds of our food, it powers our muscles.
- Heat energy, which is actually the kinetic energy of jiggling molecules, can do work in a steam engine.
- Nuclear energy in a uranium atom is the potential energy of protons whose positive electric charges explosively repel each other. (The protons are barely held together by the nuclear strong force.)

Conservation of energy is a universally confirmed law of physics: energy is never created or destroyed, only converted from one form to another. Some familiar units of energy are watt-hours of electricity, Calories in food, and tons of TNT explosive.

Mechanical work is a change in kinetic and potential energy by applying force to an object. Formally, work = (force applied) × (distance moved):

$$W = F \cdot s$$

EXAMPLE: If a 5 pound weight is lifted 10 feet, what is the work done against gravity by the lifting force? The force of gravity is measured by weight, so:

$$W = F \cdot s = (5 \text{ lb}) \cdot (10 \text{ ft}) = 50 \text{ ft-lb}.$$

Here foot-pounds (ft-lb) is another unit of energy.*

EXAMPLE: If a 5 kilogram weight is lifted 2 meters, what is the work done against gravity? This computation is more complicated, because in metric units we do not have the shortcut of using pounds as a measure for both mass (how hard it is to shove an object) and force (how hard gravity pulls the object). Rather:

$$\begin{aligned} (\text{force of gravity}) &= (\text{mass}) \cdot (\text{acceleration of gravity}) \\ F &= (5 \text{ kg}) \cdot (9.8 \text{ m/sec}^2) \\ &= 49 \text{ kg-m/sec}^2 \ = \ 49 \text{ N}. \end{aligned}$$

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*50 ft-lb ≈ 0.02 watt-hours, meaning it would take a 1 watt toy electric motor 0.02 hours, about a minute, to winch up the weight, not counting wasted energy. This is also about 0.02 food Calories: it takes a lot of lifting to burn up that candy bar (though you do a lot of extra work in moving and heating your body).

Here newtons (N) are the metric unit of force, with 1 N = 1 kg-m/sec^2 = 0.22 lbs. Rather than going through this computation, Webwork problems will usually tell you the force on a mass in newtons, or the force density in newtons per meter of rope or per cubic meter of liquid.

Now we can find:

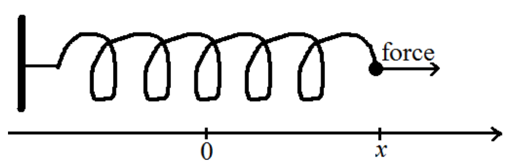
$$\begin{aligned}
\text{(work)} &= \text{(force)} \cdot \text{(distance)} \\
W &= F \cdot s \\
&= (49 \text{ N}) \cdot (2 \text{ m}) \\
&= 98 \text{ N-m} \;=\; 98 \text{ J}.
\end{aligned}$$

Here joules (J) are the metric unit of energy, with 1 joule = 1 newton-meter = 1 watt-sec = 0.74 ft-lbs.

Work against a spring force. Imagine a spring stretching along the x-axis, with its left end fixed at some negative point, and its natural length placing its right end at the equilibrium position $x = 0$. Hooke's Law says that the force required to hold the spring in an arbitrary position x is:

$$F(x) \;=\; kx,$$

where the spring constant $k > 0$ depends on the physical properties of the spring.†



Thus, if the spring is stretched out to a positive x, it requires a force in the positive direction to keep it from contracting; and if it is compressed to a negative x, it requires a force in the negative direction to keep it from bouncing back.

EXAMPLE: How much work is done in stretching a spring from $x = 2$ m to $x = 5$ m, assuming the initial condition $F(2) = 1$ N? First, since $1 = F(2) = (k)(2)$, we know that $k = \frac{1}{2}$, and $F(x) = \frac{1}{2}x$ N.

Since the force varies with x, we must split up the work W into small increments $\Delta W_1, \ldots, \Delta W_n$, whose sum approaches an integral (see Method of Slice Analysis, end of §5.2). That is, we slice the interval $x \in [2, 5]$ into n increments of length Δx, and take sample points $x_1, \ldots, x_n$. Then:

$$\begin{gathered}
\text{(work increment)} \;\approx\; \text{(force at } x_i) \cdot \text{(distance increment)} \\
\Delta W_i \;\approx\; F(x_i) \cdot \Delta x \;=\; \tfrac{1}{2} x_i \, \Delta x \,.
\end{gathered}$$

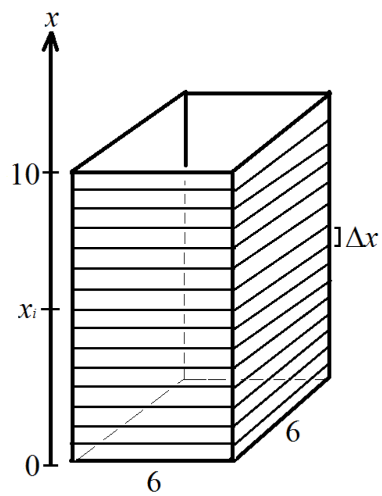
The total work is given by a Riemann sum approaching an integral:

$$W \;=\; \lim_{n\to\infty} \sum_{i=1}^{n} \Delta W_i \;=\; \lim_{n\to\infty} \sum_{i=1} \tfrac{1}{2} x_i \, \Delta x \;=\; \int_2^5 \tfrac{1}{2} x \, dx \;=\; \tfrac{1}{4} x^2 \Big|_{x=2}^{x=5} \;=\; 5.25 \,.$$

†In practice, this law only holds for x not too far from 0.

Work against gravity. An underground tank is 6 ft $\times$ 6 ft wide and 10 ft deep, with its top at ground level. How much work is done in pumping all the water up to ground level?

Draw an axis with the bottom of the tank at $x = 0$, the top at $x = 10$, and slice the tank into n thin horizontal slices, each with thickness Δx.



The force on each slice is its weight:

$$\begin{aligned}(\text{force}) = (\text{weight}) &= (\text{volume}) \cdot (\text{density of water}) \\ F &= \big((6)(6)(\Delta x)\ \text{ft}^3\big) \cdot \big(62.4\ \text{lb/ft}^3\big) \\ &= c\,\Delta x\ \text{lb}, \quad \text{where} \quad c = 2246.4\end{aligned}$$

The force F is the same for each slice, but the distance to be lifted from height x_i is $s_i = 10 - x_i$, so the increment of work is:

$$\Delta W_i = F \cdot s_i = (c\,\Delta x) \cdot (10 - x_i) = c\,(10 - x_i)\,\Delta x\,.$$

Thus the total work is:

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta W_i = \lim_{n\to\infty} \sum_{i=1}^{n} c\,(10 - x_i)\,\Delta x$$

$$= \int_0^{10} c\,(10 - x)\,dx = c\left(10x - \tfrac{1}{2}x^2\right)\Big|_{x=0}^{x=10} = 112{,}320 \text{ ft-lb.}$$

Review Problem. Compute the work to pump water from ground level to fill a spherical tank of radius 1 meter atop a 10 m tower (total height 12 m), given water density 1000 N/m^3. Choose controlling variable $y =$ height measured as $y = 0$ at the center of the tank (11 m above ground). Horizontal slice at height y has radius $x = \sqrt{1-y^2}$, force $1000\pi(1-y^2)\Delta y$; lifted by distance $11 + y$. Thus $W = \int_{-1}^{1} 1000\pi(1-y^2)(11+y)\,dy$. Multiply out and integrate to get $W = \frac{44000}{3}\pi$ N-m. Check: Identical slices at heights y and $-y$ are lifted by $11 + y$ and $11 - y$, same work as lifting both by 11, so total work equals total weight times 11: $W = 1000(\frac{4}{3}\pi 1^3)(11) = \frac{44000}{3}\pi$.

Escape velocity. How fast would a cannonball need to be shot to keep rising forever, never falling back to earth due to gravity? That is, what velocity will produce kinetic energy equal to the work in moving from the Earth's surface to an infinite height?

By Newton's Law of Universal Gravitation (and neglecting air resistance), the force on a mass of m kg at a radius r meters from Earth's center is $F_g = Cm/r^2$ newtons, where $C = 3.98 \times 10^{14}$. The work to move against this force from the surface at $r_0 = 6,370\,\text{km} = 6.37 \times 10^6$ m to $r = \infty$ is:

$$W_{\text{esc}} \;=\; \int_{r_0}^{\infty} \frac{Cm}{r^2}\,dr \;=\; Cm\left[-\frac{1}{r}\right]_{r=r_0}^{r=\infty} \;=\; Cm \lim_{r\to\infty}\left(-\frac{1}{r}+\frac{1}{r_0}\right) \;=\; \frac{Cm}{r_0}.$$

The kinetic energy of a mass m at velocity v is known to be $\frac{1}{2}mv^2$ (see below), and the escape velocity makes this equal to the above work:

$$\frac{1}{2}mv_{\text{esc}}^2 \;=\; W_{\text{esc}} \;=\; \frac{Cm}{r_0}.$$

Solving gives:

$$v_{\text{esc}} \;=\; \sqrt{\frac{2C}{r_0}} \;=\; \sqrt{\frac{2\times 3.98\times 10^{14}}{6.37\times 10^6}} \;=\; 11{,}186\,\text{m/s} \;=\; 25{,}020\,\text{mph}.$$

A high-speed tank projectile moves at less than 2000 m/s or 4500 mph, so we see why bullets and cannon balls come back down! However, any projectile which retained a speed above 25,000 mph after clearing the atmosphere would continue into space, no matter what direction it was moving (except back downward). Note that this does not apply to spaceships, whose jets constantly add acceleration and kinetic energy as they fly: the Falcon 9 only reaches 8000 mph.

Conservation of mechanical energy. Let $x(t)$ be the position at time t of a particle of unit mass $m = 1$ moving along the x-axis, pulled at each point x by a force field of strength $f(x)$. Define the *potential energy* of $f(x)$ at position $x = a$ to be the work done against the force in moving the particle from the origin to a:

$$P(a) = -\int_0^a f(x)\,dx\,.$$

(The negative sign is because the operative force acts opposite to $f(x)$.) Define the *mechanical energy* of the particle to be the potential energy of its position $x(t)$ plus its *kinetic energy* $\frac{1}{2}mv(t)^2 = \frac{1}{2}x'(t)^2$:

$$E(t) \;=\; P(x(t)) + \tfrac{1}{2}x'(t)^2.$$

The motion of $x(t)$ under force $f(x)$ is given by Newton's Second Law of Motion: force is equal to mass times acceleration, the differential equation

$$f(x(t)) = mx''(t) = x''(t).$$

Hence, the rate of change of mechanical energy is (by the Chain Rule):

$$E'(t) \;=\; P'(x(t))\,x'(t) + \tfrac{1}{2}(2x'(t)x''(t)),$$

simplifying by the First Fundamental Theorem and the Second Law to:

$$E'(t) \;=\; -f(x(t))\,x'(t) + x'(t)\,f(x(t)) \;=\; 0.$$

Since its rate of change is zero, $E(t) = E$ is constant, and energy is conserved.