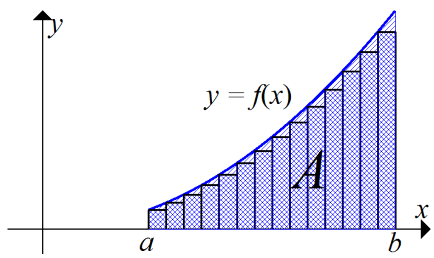
Geometry of integrals. In this section, we will learn how to compute volumes using integrals defined by slice analysis. First, we recall from Calculus I how to compute areas. Given the region under a graph $y = f(x)$ and above an interval $[a, b]$ on the x-axis, we slice up the interval into n increments (small parts) of width $\Delta x = \frac{b-a}{n}$, with division points:

$$a < a{+}\Delta x < a{+}2\Delta x < \cdots < a{+}n\Delta x = b,$$

and we take sample points $x_1, \ldots, x_n$, one in each increment. This slices the area into increments $\Delta A_1, \ldots, \Delta A_n$, each approximately a rectangle:



The area of an increment is approximately:

$$\Delta A_i \;\approx\; (\text{height})\times(\text{width}) \;=\; f(x_i)\,\Delta x\,,$$

so the total area is the Riemann sum (see §4.1 Pt 1):

$$A \;=\; \sum_{i=1}^{n} \Delta A_i \approx \sum_{i=1}^{n} f(x_i)\Delta x\,.$$

Taking the limit of very many, very thin slices defines the integral, which computes the exact area:

$$A \;=\; \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \;=\; \int_a^b f(x)\,dx\,.$$

This only gives approximate numerical answers to our question, though, unless we can determine the limit, which is very difficult directly (§4.1 Pt 2).

So far, there is no really surprising idea here: for thousands of years, mathematicians used similar methods to compute areas, with very limited success. The genius of Newton shines in three ideas which solve the problem, easily computing almost any area (§4.3). The first idea is to make area into a function: $A(x) = \int_a^x f(t)\,dt$ is the area over a variable interval $t \in [a, x]$. Second, we find that $A'(x) = f(x)$: amazingly, the area function is an antiderivative of $f(x)$, because the rate of change of area is the height of the graph at the moving endpoint. That is, as x moves rightward, $A(x)$

Notes by Peter Magyar `magyar@math.msu.edu` increases quickly or slowly depending on how high $f(x)$ is: this is the First Fundamental Theorem.

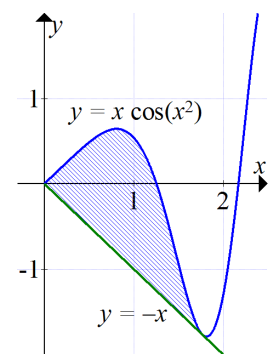
The third and clinching idea is the algebra of differentiation. If we can reverse the derivative rules to find another antiderivative, a known function $F(x)$ with $F'(x) = f(x)$, then the Uniqueness Theorem (§3.2) tells us that the two antiderivatives are the same, except for adding some constant: $A(x) = \int_a^x f(t)\,dt \;=\; F(x) + C$. Since $0 = A(a) = F(a) + C$, we must have $C = -F(a)$, so $A(x) = F(x) - F(a)$, the Second Fundamental Theorem:

$$A = A(b) = \int_a^b f(x)\,dx \;=\; F(b) - F(a).$$

That is, if $f(x)$ is the *rate of change* of $F(x)$, so that $\Delta F_i = f(x_i)\Delta x$ are the small changes (increments) of $F(x)$, then the integral is the *total change* in $F(x)$ over the interval $[a, b]$.*

These three brilliant ideas, working perfectly together, make it easy to compute the areas of most shapes, provided we are skillful enough at reversing derivative rules to find indefinite integrals.

EXAMPLE: Find the area of the leftmost region enclosed below the curve $y = x\cos(x^2)$ and above the line $y = -x$.



The left corner of the region is at $x = 0$. To find the right corner, we set $x\cos(x^2) = -x$ and get $x(1+\cos(x^2)) = 0$, whose smallest positive solution is $x = \sqrt{\pi} \approx 1.77$.

To compute the area, we consider thin slices from the top curve to the bottom line, with height $x\cos(x^2) - (-x) = x\cos(x^2) + x$, and take a limit of Riemann sums to get:

$$A \;=\; \int_0^{\sqrt{\pi}} \big(x\cos(x^2) + x\big)\,dx \;=\; \int_0^{\sqrt{\pi}} x\cos(x^2)\,dx \;+\; \int_0^{\sqrt{\pi}} x\,dx.$$

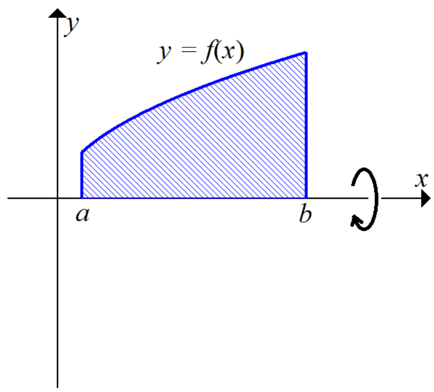
We can evaluate the first term using the substitution $u = x^2$, $du = 2x\,dx$, so $\int x\cos(x^2)\,dx = \frac{1}{2}\int \cos(x^2)\,2x\,dx = \frac{1}{2}\int \cos(u)\,du = \frac{1}{2}\sin(u) = \frac{1}{2}\sin(x^2)$.

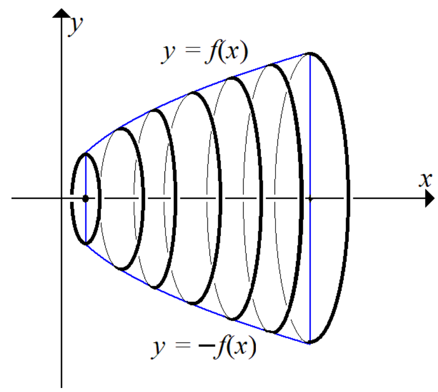
$$A \;=\; \Big[\tfrac{1}{2}\sin(x^2) + \tfrac{1}{2}x^2\Big]_{x=0}^{x=\sqrt{\pi}} \;=\; \tfrac{1}{2}\pi \;\approx\; 1.57\,.$$

*The *indefinite integral* $\int f(x)\,dx = F(x) + C$ is the general antiderivative function, also called the primitive function. The *definite integral* $\int_a^b f(x) = F(b) - F(a)$ is a number.

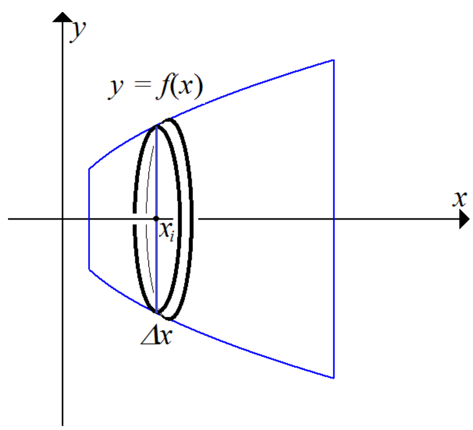
That looks about right from the picture. Not a result you could get from elementary geometry!

Solids of revolution. The integral is a very powerful tool to compute the total of many small parts, such as the thin slices which fill up an area. A similar method allows us to compute volumes. We start with a *solid of revolution* which is formed by rotating a region in the plane around the x-axis, sweeping out a kind of barrel shape.





The total volume is the sum of n disk slices at sample points $x_1, \ldots, x_n$ in $[a, b]$, each with radius $f(x_i)$ and thickness $\Delta x = \frac{b-a}{n}$.



The volume of each slice is approximately:

$$\Delta V_i \;\approx\; (\text{circle area}) \times (\text{thickness}) \;=\; \pi f(x_i)^2 \, \Delta x \, ,$$

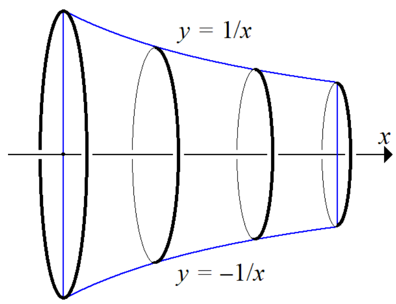
and taking the limit as $n \to \infty$ gives the exact volume as an integral:

$$V \;=\; \lim_{n\to\infty} \sum_{i=1}^{n} \pi f(x_i)^2 \, \Delta x \;=\; \int_a^b \pi f(x)^2 \, dx \, .$$

Once we express our answer as an integral, we no longer consider its geometric motivation: finding an antiderivative and determining the value is a purely algebraic problem.

This analysis should not be surprising: we expect our answer to converge to an integral as soon as we can slice up the problem into small increments.

EXAMPLE: Find the volume of the trumpet solid obtained by rotating around the x-axis the region defined by: $0 \le y \le \frac{1}{x}$ and $1 \le x \le 2$. This is the solid of revoltion of the curve $y = \frac{1}{x}$ over the interval $x \in [a, b]$.

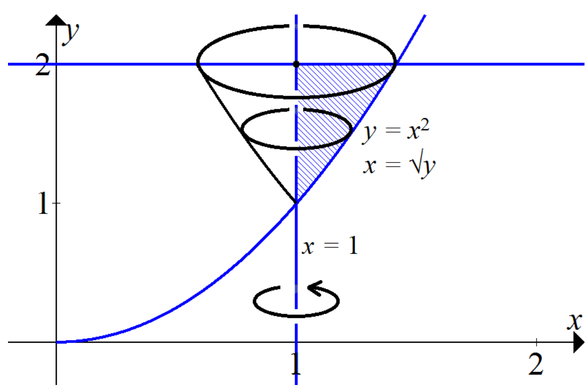


Applying our formula:

$$V \;=\; \int_1^2 \pi \left(\tfrac{1}{x}\right)^2 dx \;=\; \int_1^2 \pi x^{-2}\, dx$$

$$= \;\left[-\pi x^{-1}\right]_{x=1}^{x=2} \;=\; (-\pi 2^{-1}) - (-\pi 1^{-1}) \;=\; \tfrac{\pi}{2}\,.$$

EXAMPLE: Find the volume of the solid obtained by rotating the region defined by $x^2 \le y \le 2$ and $x \ge 1$, around the vertical axis $x = 1$.

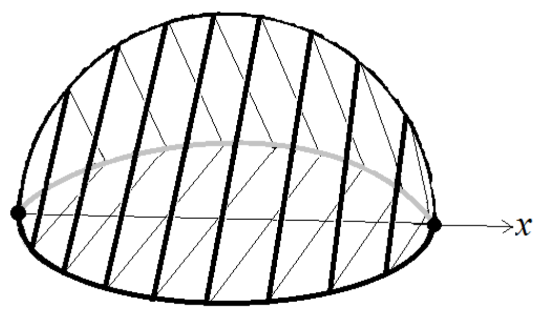


The setup is different from the x-axis rotation, so we must repeat our slice analysis. Since we rotate around the vertical line $x = 1$, we should take horizontal slices positioned by their height y, and we should write our region as lying next to the interval $y \in [1, 2]$ between the curves $x = 1$ and $x = \sqrt{y}$.

Our slices are thin horizontal disks at sample heights $y_1, \ldots, y_n$ in the interval $y \in [1, 2]$. The thickness of each disk is Δy. The radius at height y_i is the horizontal distance from the axis $x = 1$ to the curve $x = \sqrt{y}$; this distance is $\sqrt{y_i}-1$. The volume of each disk is $\Delta V_i \approx \pi(\sqrt{y_i}-1)^2\,\Delta y$, and:

$$V \;=\; \int_1^2 \pi(\sqrt{y}-1)^2\; dy \;=\; \int_1^2 \pi(y-2\sqrt{y}+1)\,dy \;=\; \left[\pi\left(\tfrac{1}{2}y^2-\tfrac{4}{3}y^{3/2}+y\right)\right]_{y=1}^{y=2} \;=\; \pi\left(\tfrac{23}{6}-\tfrac{8\sqrt{2}}{3}\right).$$

Solids with specified cross sections. Let us consider a solid rather like the sail of the man-of-war jellyfish. It has a base outlined by the circle $x^2+y^2=1$, and its vertical cross section over each line $x=c$ is an isosceles right triangle, with height equal to half its base.



The slices are defined over $x\in[-1,1]$, with each slice at $x=x_i$ having thickness Δx and area $\frac{1}{2}(\text{base})\times(\text{height}) = \frac{1}{2}\left(2\sqrt{1-x_i^2}\,\right)\left(\sqrt{1-x_i^2}\,\right)$. Thus:

$$V \;=\; \int_{-1}^{1} \tfrac{1}{2}\left(2\sqrt{1-x^2}\right)\left(\sqrt{1-x^2}\right)\,dx \;=\; \int_{-1}^{1} 1-x^2\,dx \;=\; \left[x-\tfrac{1}{3}x^3\right]_{x=-1}^{x=1} \;=\; \tfrac{4}{3}.$$

Method of slice analysis to compute size. We have now seen several cases where we compute the size or bulk of geometric objects. Let S denote any measure of the size of an object: length, area, volume, mass, etc.

1. Cut the object into slices whose position is determined by some variable $x\in[a,b]$.

2. Mark off the interval $[a,b]$ into n increments of width $\Delta x = \frac{b-a}{n}$, each with a sample point x_i. This splits the object into n slices, and summing up their sizes gives the total size: $S=\sum_{i=1}^n \Delta S_i$.

3. Because the slice at x_i is so thin, we can find a good approximation of its size by some simple formula of the form $\Delta S_i \approx f(x_i)\,\Delta x$.

4. Taking $n\to\infty$ and $\Delta x\to 0$, the approximation becomes exact:

$$S \;=\; \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x \;=\; \int_a^b f(x)\,dx\,.$$

5. Having expressed $S=\int_a^b f(x)\,dx$, we evaluate this integral by algebraic or numerical techniques.

CHALLENGE PROBLEM: Consider the solid of revolution around the x-axis of the curve $y = f(x)$ over the interval $x \in [a, b]$. Explain the following formula for the surface area of this solid:

$$S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\, dx\,.$$

Hint: $\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + (\frac{\Delta y}{\Delta x})^2}\ \Delta x \approx \sqrt{1 + (\frac{dy}{dx})^2}\ \Delta x.$