We have one more important test for convergence of an infinite series \( \sum_{n=1}^{\infty} a_n \). This test does not require us to choose a comparison series; instead, we test the ratio of each term \( a_n \) compared to the next term \( a_{n+1} \).

**Ratio Test:** Suppose \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \).

- If \( L < 1 \), then \( \sum_{n=1}^{\infty} a_n \) converges.
- If \( L > 1 \), then \( \sum_{n=1}^{\infty} a_n \) diverges.
- If \( L = 1 \), then this test fails to determine convergence.

**Proof:** Assuming \( a_n > 0 \), the limit \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \) means that, for any small number \( \epsilon > 0 \), we can take a starting point \( N \) so that for all \( n \geq N \), we have:

\[
L - \epsilon \leq \frac{a_{n+1}}{a_n} \leq L + \epsilon
\]

Iterating this inequality gives: \( c_1(L-\epsilon)^n \leq a_n \leq c_2(L+\epsilon)^n \) for some constants \( c_1, c_2 \).

If \( L < 1 \), we take \( \epsilon \) small enough that \( L+\epsilon < 1 \), and we compare \( \sum a_n \) to the convergent ceiling series \( \sum c_2(L+\epsilon)^n \). If \( L > 1 \), we take \( \epsilon \) small enough that \( L-\epsilon > 1 \), and we compare \( \sum a_n \) to the divergent floor series \( \sum c_2(L-\epsilon)^n \). If \( L = 1 \), adding any \( \epsilon \) produces a divergent ceiling, and subtracting any \( \epsilon \) produces a convergent floor, neither of which would constrain the original series. Finally, for the general case where the \( a_n \)'s may be positive or negative, the above argument shows \( \sum |a_n| \) converges, which implies \( \sum a_n \) converges by §11.6 Part II. Q.E.D.

The Ratio Test is most useful when \( a_n \) is a product of a growing number of factors, which will mostly cancel out in \( \frac{a_{n+1}}{a_n} \).

**EXAMPLE:** Determine the convergence of \( \sum_{n=1}^{\infty} \frac{n^2}{2^n} \).

We did this one in §11.4 by finding a tricky comparison series. The Ratio Test naturally applies here, because \( a_n = \frac{n^2}{2^n} = (n)(n)(\frac{1}{2})\cdots(\frac{1}{2}) \) has more and more factors as \( n \) gets larger. We have:

\[
L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)^2}{2^{n+1}} \right| / \left| \frac{n^2}{2^n} \right| = \lim_{n \to \infty} \frac{(n+1)^2}{n^2} / \left( \frac{2}{n} \right) = 1 / 2.
\]

Since \( L = \frac{1}{2} < 1 \), the Test shows \( \sum a_n \) converges.

**EXAMPLE:** Determine the convergence of \( \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{n!} \), where \( x \) is a given number and we use the factorial notation \( n! = (n)(n-1)(n-2)\cdots(1) \). Again, the terms have a large number of factors, so we use the Ratio Test:

\[
L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{x^{2(n+1)}}{(n+1)!} / \left( \frac{x^{2n}}{n!} \right) = \lim_{n \to \infty} \frac{x}{n+1} = 0.
\]

Since \( L = 0 < 1 \), the Test shows \( \sum a_n \) converges.

*Specifically: \( a_n \leq a_{n-1}(L+\epsilon) \leq a_{n-2}(L+\epsilon)^2 \leq \cdots \leq a_N(L+\epsilon)^{n-N} = \frac{a_N}{(L+\epsilon)^N}(L+\epsilon)^n \).