Convergence and divergence. We continue to discuss convergence tests: ways to tell if a given series $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$ converges (to a finite value), or diverges (to infinity or by oscillating).* So far, we know convergence for two kinds of standard series:

- Geometric series: $\sum_{n=1}^{\infty} cr^{n-1}$ converges to $\frac{c}{1-r}$ if $|r| < 1$, diverges if $|r| \geq 1$.
- Standard p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, and diverges if $p \leq 1$.

In this section, we test convergence of a complicated series $\sum a_n$ by comparing it to a simpler one (such as the above): a convergent ceiling $\sum c_n$, or a divergent floor $\sum d_n$.

Direct Comparison Test: Let M be a positive integer starting point.

- If $0 \leq a_n \leq c_n$ for $n \geq M$, and $\sum\limits_{n=1}^{\infty} c_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges.
- If $a_n \geq d_n \geq 0$ for $n \geq M$, and $\sum\limits_{n=1}^{\infty} d_n$ diverges, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

These results are clear, since the series $\sum_{n=1}^{\infty} a_n$ is term-by-term smaller or larger than its comparison series, except possibly the first $M-1$ terms.†

Example: Determine convergence of: $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}+1}$. We have:

$$a_n = \frac{n-1}{n^2\sqrt{n}+1} \quad \leq \quad c_n = \frac{n}{n^2\sqrt{n}} = \frac{1}{n^{3/2}} \qquad \text{for } n \geq 1,$$

since on the left the numerator is smaller and the denominator is larger than on the right. The comparison series $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a standard p-series which converges, so $\sum_{n=1}^{\infty} a_n$ also converges.

Example: Determine the convergence of: $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3n+\sin(n)}}{3^n + 4n^2}$.

As a rough guess, we ignore the lower-order terms in numerator and denominator to compare with $\frac{2^{3n}}{3^n} = \left(\frac{8}{3}\right)^n$, which makes a divergent geometric series, so our series a_n should also diverge. However, it is not clear that a_n is really larger than this comparison series, so we cannot use $d_n = \left(\frac{8}{3}\right)^n$ as a divergent floor for a_n in the second part of the Comparison Test.

We want to produce a fractional d_n from our a_n by making the numerator smaller and the denominator larger. To bound the numerator: $2^{3n+\sin(n)} = 2^{3n}2^{\sin(n)} \geq$

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*A general divergent series might oscillate up and down forever, but a positive series (with $a_n \geq 0$) either levels off to a finite value, or diverges to infinity.

†Here we use the completeness axiom of real analysis, which states that if a series of partial sums has an upper bound, $s_N = \sum_{n=1}^{N} a_n < B$ for all N, then the least upper bound $L = \lim\limits_{N\to\infty} s_N$ exists.

$2^{3n}2^{-1}$. To bound the denominator, we take an exponential function with a slightly larger base: we can check that $4^n \geq 3^n + 4n^2$ for all $n \geq 3$. Thus:

$$a_n = \frac{2^{3n+\sin(n)}}{3^n + n^2} \quad \geq \quad d_n = \frac{2^{3n}2^{-1}}{4^n} = \tfrac{1}{2}2^n \qquad \text{for } n \geq 3.$$

Note that we only need the inequality for all large n: the first couple of terms a_1, a_2 make no difference to the convergence or divergence. Since $\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty} \frac{1}{2}2^n$ is a divergent geometric series, the orginal $\sum_{n=1}^{\infty} a_n$ also diverges.

EXAMPLE: Determine convergence of: $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^3-20}$.

Again, we estimate this sequence by its leading terms: $\sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent standard p-series. However, $a_n = \frac{n+1}{n^3-20} > \frac{n}{n^3}$, so we cannot use $c_n = \frac{n}{n^3}$ as a convergent ceiling for a_n in the first part of the Test.

However, we should have:

$$a_n = \frac{n+1}{n^3-20} \;\leq\; c_n = 2\frac{n}{n^3} \qquad \text{for } n \text{ large enough.}$$

How large does n need to be to make this inequality valid? Let us check:

$$\frac{n+1}{n^3-20} \leq \frac{2}{n^2} \quad \Longleftarrow \quad 0 < n^2(n{+}1) \leq 2(n^3{-}20) \quad \Longleftrightarrow \quad 40 \leq n^2(n{-}1) \quad \Longleftarrow \quad n \geq 4\,.$$

Thus, we have:

$$a_n = \frac{n+1}{n^3-20} \;\leq\; c_n = \frac{2}{n^2} \qquad \text{for } n \geq 4,$$

where $\sum_{n=1}^{\infty} \frac{2}{n^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so the original $\sum_{n=1}^{\infty} a_n$ also converges.

EXAMPLE: Consider any infinite decimal:

$$s \;=\; 0.d_1 d_2 d_3 \cdots \;=\; \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots \;=\; \sum_{n=1}^{\infty} \frac{d_n}{10^n}\,,$$

where $0 \leq d_n \leq 9$ are *any* decimal digits. Does this series always converge, so that the infinite decimal represents a real number, or could a bad choice of digits define a meaningless decimal?

In fact, we can compare $0 \leq \frac{d_n}{10^n} \leq \frac{9}{10^n}$, since each digit is at most 9. The ceiling is a convergent geometric series: $\sum_{n=1}^{\infty} \frac{9}{10^n} = \sum_{n=1}^{\infty} \frac{9}{10}\left(\frac{1}{10}\right)^{n-1} = \frac{9}{10}\frac{1}{1-\frac{1}{10}} = 1$, so the original decimal sequence also converges. *Any* infinite decimal represents a number.

Limit Comparison Test. Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ with $0 < L < \infty$.

- If $\sum\limits_{n=1}^{\infty} b_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges.
- If $\sum\limits_{n=1}^{\infty} b_n$ diverges, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Proof: $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ means that, for any small $\epsilon > 0$, we can take a starting point N so that for all $n \geq N$, we have:

$$L{-}\epsilon \;\leq\; \tfrac{a_n}{b_n} \;\leq\; L{+}\epsilon \qquad \text{and} \qquad (L{-}\epsilon)b_n \;\leq\; a_n \;\leq\; (L{+}\epsilon)b_n\,.$$

Taking ϵ small enough that $L{\pm}\epsilon > 0$, we can prove convergence or divergence by taking $c_n = (L{+}\epsilon)b_n$ or $d_n = (L{-}\epsilon)b_n$ in the Direct Comparison Test.

EXAMPLE: We redo $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^3-20}$. Now we can immediately compare with $b_n = \dfrac{n}{n^3}$:

$$\frac{a_n}{b_n} \;=\; \frac{n+1}{n^3-20} \Big/ \frac{n}{n^3} \;=\; \frac{n+1}{n} \Big/ \frac{n^3-20}{n^3} \;=\; \frac{1+\frac{1}{n}}{1-\frac{20}{n^3}}\,.$$

Taking $n \to \infty$ gives $L = 1$. Since this satisfies $0 < L < \infty$, and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent standard p-series, the original series $\sum_{n=1}^{\infty} a_n$ also converges.

Extended Limit Comparison Test. In the case where $\lim_{n\to\infty} \frac{a_n}{b_n} = L = 0$, we have a_n much smaller than b_n, so if $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$. Similarly, in the case where $\lim_{n\to\infty} \frac{a_n}{b_n} = L = \infty$, we have a_n much larger than b_n, so if $\sum_{n=1}^{\infty} b_n$ diverges, then so does $\sum_{n=1}^{\infty} a_n$.

EXAMPLE: Determine the convergence of: $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

Since n^2 is negligeable compared to the exponential growth of 2^n, we could roughly estimate this by $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$, a convergent geometric series, so the original series should converge.

However, taking the Limit Comparison Test with this $b_n = \frac{1}{2^n}$ gives $L = \infty$, since $a_n = \frac{n^2}{2^n}$ is much *larger* than b_n. Thus this comparison fails: b_n is a convergent floor for a_n, and we can't tell whether $\sum a_n$ converges or diverges.

Let us instead take a slightly larger, but still convergent, comparison: $b_n = \left(\frac{3}{4}\right)^n$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} \;=\; \lim_{n\to\infty} \frac{n^2\left(\frac{1}{2}\right)^n}{\left(\frac{3}{4}\right)^n} \;=\; \lim_{n\to\infty} n^2\left(\tfrac{2}{3}\right)^n \;=\; 0\,,$$

as we could prove by L'Hôpital's Rule. Thus $a_n = \frac{n^2}{2^n}$ becomes much *smaller* than b_n, and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$ is a convergent ceiling for $\sum_{n=1}^{\infty} a_n$, which therefore must also converge.