Math 133  
Comparison Tests  
Stewart §11.4

Convergence and divergence. We continue to discuss convergence tests: ways to tell if a given series \( \sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} \sum_{n=1}^{N} a_n \) converges (to a finite value), or diverges (to infinity or by oscillating).\(^*\) So far, we know convergence for two kinds of standard series:

- Geometric series: \( \sum_{n=1}^{\infty} c r^{n-1} \) converges to \( c / (1-r) \) if \( |r| < 1 \), diverges if \( |r| \geq 1 \).
- Standard \( p \)-series: \( \sum_{n=1}^{\infty} 1/n^p \) converges if \( p > 1 \), and diverges if \( p \leq 1 \).

In this section, we test convergence of a complicated series \( \sum a_n \) by comparing it to a simpler one (such as the above): a convergent ceiling \( \sum c_n \), or a divergent floor \( \sum d_n \).

Direct Comparison Test: Let \( N \) be a positive integer starting point.

- If \( 0 \leq a_n \leq c_n \) for \( n \geq N \), and \( \sum_{n=1}^{\infty} c_n \) converges, then \( \sum_{n=1}^{\infty} a_n \) converges.
- If \( a_n \geq d_n \geq 0 \) for \( n \geq N \), and \( \sum_{n=1}^{\infty} d_n \) diverges, then \( \sum_{n=1}^{\infty} a_n \) diverges.

These results are clear, since the series \( \sum_{n=1}^{\infty} a_n \) is term-by-term smaller or larger than its comparison series.\(^\dagger\)

Example: Determine convergence of: \( \sum_{n=1}^{\infty} \frac{n - 1}{n^2 \sqrt{n} + 1} \). We have:

\[
a_n = \frac{n - 1}{n^2 \sqrt{n} + 1} \leq c_n = \frac{n}{n^2 \sqrt{n}} = \frac{1}{n^{3/2}} \quad \text{for } n \geq 1,
\]

since on the left the numerator is smaller and the denominator is larger than on the right. The comparison series \( \sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \) is a standard \( p \)-series which converges, so \( \sum_{n=1}^{\infty} a_n \) also converges.

Example: Determine the convergence of: \( \sum_{n=1}^{\infty} \frac{2^{3n+\sin(n)}}{3^n + 4n^2} \).

As a rough guess, we ignore the lower-order terms in numerator and denominator to compare with \( \frac{2^{3n}}{3^n} = \left(\frac{8}{3}\right)^n \), which makes a divergent geometric series, so our series \( a_n \) should also diverge. However, it is not clear that \( a_n \) is really larger than this comparison series, so we cannot use \( d_n = \left(\frac{8}{3}\right)^n \) as a divergent floor for \( a_n \) in the second part of the Comparison Test.

We want to produce a fractional \( d_n \) from our \( a_n \) by making the numerator smaller and the denominator larger. To bound the numerator: \( 2^{3n+\sin(n)} \leq \frac{2^{3n+1}}{3^n} \). \(^*\)

\(\dagger\)Here we use the completeness axiom of real analysis, which states that if a series of partial sums has an upper bound, \( s_N = \sum_{n=1}^{N} a_n < B \) for all \( N \), then the least upper bound \( L = \lim_{N \to \infty} s_N \) exists.

\(\ast\)A general divergent series might oscillate up and down forever, but a positive series (with \( a_n \geq 0 \)) either levels off to a finite value, or diverges to infinity.
\[2^{3n}2^{-1}\]. To bound the denominator, we take an exponential function with a slightly larger base: we can check that \(4^n \geq 3^n + 4n^2\) for all \(n \geq 3\). Thus:

\[
a_n = \frac{2^{3n} \sin(n)}{3^n + n^2} \geq d_n = \frac{2^{3n}2^{-1}}{4^n} = \frac{1}{2}2^n \quad \text{for } n \geq 3.
\]

Note that we only need the inequality for all large \(n\): the first couple of terms \(a_1, a_2\) make no difference to the convergence or divergence. Since \(\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty} \frac{1}{2}2^n\) is a divergent geometric series, the orginal \(\sum_{n=1}^{\infty} a_n\) also diverges.

**EXAMPLE:** Determine convergence of:

\[
\sum_{n=1}^{\infty} \frac{n + 1}{n^3 - 20}.
\]

Again, we estimate this sequence by its leading terms: \(\sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}\), which is a convergent standard \(p\)-series. However, \(a_n = \frac{n + 1}{n^3 - 20} > \frac{n}{n^3}\), so we cannot use \(c_n = \frac{n}{n^3}\) as a convergent ceiling for \(a_n\) in the first part of the Test.

However, we should have:

\[
a_n = \frac{n + 1}{n^3 - 20} \leq c_n = \frac{2}{n^2} \quad \text{for } n \text{ large enough}.
\]

How large does \(n\) need to be to make this inequality valid? Let us check:

\[
\frac{n + 1}{n^3 - 20} \leq \frac{2}{n^2} \iff 0 < n^2(n+1) \leq 2(n^3 - 20) \iff 40 \leq n^2(n-1) \iff n \geq 4.
\]

Thus, we have:

\[
a_n = \frac{n + 1}{n^3 - 20} \leq c_n = \frac{2}{n^2} \quad \text{for } n \geq 4,
\]

where \(\sum_{n=1}^{\infty} \frac{2}{n^2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}\) converges, so the original \(\sum_{n=1}^{\infty} a_n\) also converges.

**EXAMPLE:** Consider any infinite decimal:

\[
s = 0.d_1d_2d_3\cdots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots = \sum_{n=1}^{\infty} \frac{d_n}{10^n},
\]

where \(0 \leq d_n \leq 9\) are any decimal digits. Does this series always converge, so that the infinite decimal represents a real number, or could a bad choice of digits define a meaningless decimal?

In fact, we can compare \(0 \leq \frac{d_n}{10^n} \leq \frac{9}{10^n}\), since each digit is at most 9. The ceiling is a convergent geometric series: \(\sum_{n=1}^{\infty} \frac{9}{10^n} = \sum_{n=1}^{\infty} \frac{9}{10} \left(\frac{1}{10}\right)^{n-1} = \frac{9}{10} \frac{1}{1 - \frac{1}{10}} = 1\), so the original decimal sequence also converges. Any infinite decimal represents a number.

**Limit Comparison Test.** Suppose \(\lim_{n \to \infty} \frac{a_n}{b_n} = L\) with \(0 < L < \infty\).

- If \(\sum_{n=1}^{\infty} b_n\) converges, then \(\sum_{n=1}^{\infty} a_n\) converges.
- If \(\sum_{n=1}^{\infty} b_n\) diverges, then \(\sum_{n=1}^{\infty} a_n\) diverges.
Proof: \( \lim_{n \to \infty} \frac{a_n}{b_n} = L \) means that, for any small \( \epsilon > 0 \), we can take a starting point \( N \) so that for all \( n \geq N \), we have:

\[
L - \epsilon \leq \frac{a_n}{b_n} \leq L + \epsilon \quad \text{and} \quad (L-\epsilon)b_n \leq a_n \leq (L+\epsilon)b_n.
\]

Taking \( \epsilon \) small enough that \( L \pm \epsilon > 0 \), we can prove convergence or divergence by taking \( c_n = (L+\epsilon)b_n \) or \( d_n = (L-\epsilon)b_n \) in the Direct Comparison Test.

Example: We redo \( \sum_{n=1}^{\infty} \frac{n+1}{n^3 - 20} \). Now we can immediately compare with \( b_n = \frac{n}{n^3} \):

\[
\frac{a_n}{b_n} = \frac{n+1}{n^3 - 20} \cdot \frac{n}{n^3} = \frac{n + 1}{n} \cdot \frac{n^3 - 20}{n^3} = \frac{1 + \frac{1}{n}}{1 - \frac{20}{n^2}}.
\]

Taking \( n \to \infty \) gives \( L = 1 \). Since this satisfies \( 0 < L < \infty \), and \( \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \) is a convergent standard \( p \)-series, the original series \( \sum_{n=1}^{\infty} a_n \) also converges.

Extended Limit Comparison Test. In the case where \( \lim_{n \to \infty} \frac{a_n}{b_n} = L = 0 \), we have \( a_n \) much smaller than \( b_n \), so if \( \sum_{n=1}^{\infty} b_n \) converges, then so does \( \sum_{n=1}^{\infty} a_n \). Similarly, in the case where \( \lim_{n \to \infty} \frac{a_n}{b_n} = L = \infty \), we have \( a_n \) much larger than \( b_n \), so if \( \sum_{n=1}^{\infty} b_n \) diverges, then so does \( \sum_{n=1}^{\infty} a_n \).

Example: Determine the convergence of: \( \sum_{n=1}^{\infty} \frac{n^2}{2^n} \).

Since \( n^2 \) is negligible compared to the exponential growth of \( 2^n \), we could roughly estimate this by \( \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \), a convergent geometric series, so the original series should converge.

However, taking the Limit Comparison Test with this \( b_n = \frac{1}{2^n} \) gives \( L = \infty \), since \( a_n = \frac{n^2}{2^n} \) is much larger than \( b_n \). Thus this comparison fails: \( b_n \) is a convergent floor for \( a_n \), and we can’t tell whether \( \sum a_n \) converges or diverges.

Let us instead take a slightly larger, but still convergent, comparison: \( b_n = \left(\frac{3}{4}\right)^n \):

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^2 \left(\frac{1}{2}\right)^n}{\left(\frac{3}{4}\right)^n} = \lim_{n \to \infty} n^2 \left(\frac{2}{3}\right)^n = 0,
\]

as we could prove by L’Hôpital’s Rule. Thus \( a_n = \frac{n^2}{2^n} \) becomes much smaller than \( b_n \), and \( \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \) is a convergent ceiling for \( \sum_{n=1}^{\infty} a_n \), which therefore must also converge.