Integral Test

Series and integrals. Our goal for infinite series is to express complicated quantities as infinite series of simple terms, so that finite partial sums approximate the original quantity as accurately as we like. We will not have significant tools to achieve this until §11.9.

For now (11.3-7), we concentrate on a more elementary question: when does a given series converge to some finite value? For example, we have seen the nth Term Vanishing Test: the series must diverge if the the terms do not approach zero, $\lim_{n\to\infty} a_n \neq 0$. A more subtle and powerful convergence test comes from comparing the sum of a series to the area under a curve $y = f(x)$ passing through each point (n, a_n).

> *Integral Test:* Suppose the function $f(x)$ is continuous, positive, and decreasing on the interval $x \in [1, \infty)$, and that $a_n = f(n)$. We compare the improper integral $\int_1^\infty f(x)\,dx$ with the infinite series $\sum_{n=1}^\infty a_n$.
>
> - If $\int_1^\infty f(x)\,dx$ diverges, then $\sum_{n=1}^\infty a_n$ also diverges.
> - If $\int_1^\infty f(x)\,dx$ converges, then $\sum_{n=1}^\infty a_n$ also converges.

Divergent case. Consider $\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty \frac{1}{n}$. Then the function $f(x) = \frac{1}{x}$ has $a_n = f(n)$, and for $x \in [1, \infty)$, this function is:

- continuous, since its only vertical asymptote is $x = 0$, outside $x \in [1, \infty)$;
- positive, since $x \geq 1$ implies $\frac{1}{x} > 0$;
- decreasing, since its derivative is negative,* $f'(x) = -\frac{1}{x^2} < 0$.

Thus, we can apply the Integral Test to compare the infinite series with the improper integral $\int_1^x f(x)\,dx$. We compute:

$$\int_1^\infty \tfrac{1}{x}\,dx \;=\; \lim_{n\to\infty} \int_1^n \tfrac{1}{x}\,dx \;=\; \lim_{n\to\infty} \ln(x)\Big|_{x=1}^{x=n} \;=\; \lim_{n\to\infty} \ln(n) - \ln(1) \;=\; \infty.$$

Since the integral diverges, the series $\sum_{n=1}^\infty a_n$ must also diverge.

The Integral Test is best understood geometrically. The value of the series is the total area under the bar-graph of $\{a_n\}$, where we draw the bar at height a_n above the interval $x \in [n, n+1]$.

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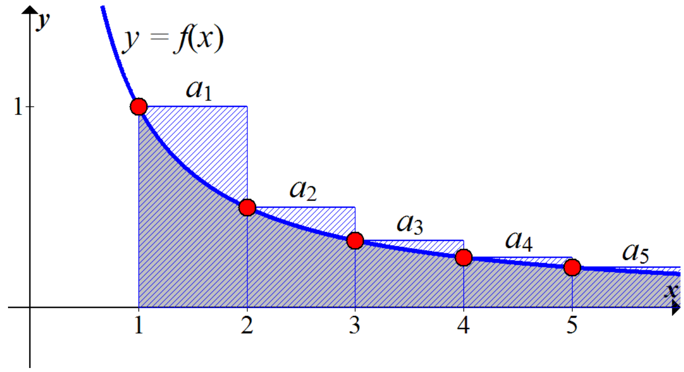
*See §3.3 Derivatives and Graphs, in the Math 132 Lecture Notes



The integral is the area of the region under $y = f(x)$ and above $x \in [1, \infty)$. But the bar graph completely contains the integral region, so:

$$\sum_{n=1}^{\infty} a_n \;\geq\; \int_1^{\infty} f(x)\, dx\,.$$

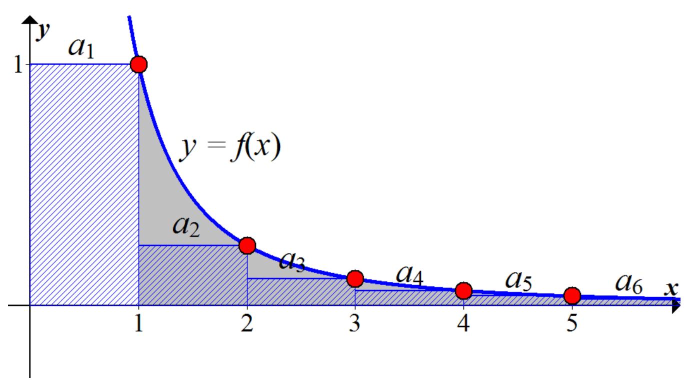
Since the integral diverges to infinity, getting larger and larger with no bound as we add area on the right, the series must also diverge as we add more terms.

Convergent case. Now consider $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which has $a_n = f(n)$ for $f(x) = \frac{1}{x^2}$, which is again a continuous, positive, decreasing function for $x \in [1, \infty)$. Thus we can apply the Integral Test to compare the infinte series with the improper integral $\int_1^x f(x)\, dx$:

$$\int_1^{\infty} \tfrac{1}{x^2}\, dx \;=\; \lim_{n\to\infty} \int_1^n \tfrac{1}{x^2}\, dx \;=\; \lim_{n\to\infty} -\tfrac{1}{x}\Big|_{x=1}^{x=n} \;=\; \lim_{n\to\infty} -\tfrac{1}{n} - (-\tfrac{1}{1}) \;=\; 1.$$

Since the integral converges, the series $\sum_{n=1}^{\infty} a_n$ must also converge.

Again, we can understand this geometrically. The value of the series is the total area under the bar-graph of $\{a_n\}$, but this time we draw the bar at height a_n above the interval $x \in [n{-}1, n]$, shifted left from our previous method. Notice that the heights of the bars are $a_n = \frac{1}{n^2}$, much lower than the previous example with $a_n = \frac{1}{n}$, so this area has a better chance of converging to a finite value.



Clearly, the part of the bar-graph after a_2 is contained in the integral region under

$y = f(x)$ and above $x \in [1, \infty)$. Thus:

$$\sum_{n=2}^{\infty} a_n \;\leq\; \int_1^{\infty} f(x)\,dx\,,$$

$$\sum_{n=1}^{\infty} a_n \;=\; a_1 + \sum_{n=2}^{\infty} a_n \;\leq\; a_1 + \int_1^{\infty} f(x)\,dx \;=\; 2\,.$$

We conclude not only that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is finite, but that it is at most 2.†

Examples.

- *Standard p-series.* The above reasoning is easily generalized to show:

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges if } p > 1, \text{ diverges if } p \leq 1.}$$

- Determine the convergence of:

$$\sum_{n=1}^{\infty} a_n \;=\; \sum_{n=1}^{\infty} \frac{1}{(2n-5)(2n-7)}\,.$$

In this case, the function $f(x) = \frac{1}{(2x-5)(2x-7)}$ has vertical asymptotes at $x = \frac{5}{2}, \frac{7}{2}$: it is not continuous, or positive, or decreasing on $x \in [1, \infty)$, so the Integral Test does not immediately apply.

However, to the right of the asympotes, for $x \in [4, \infty)$, the function is continuous; it is positive, since $2x - 5 > 0$ and $2x - 7 > 0$ for $x \geq 4$; and it is decreasing, since the derivative $f'(x) = -\frac{8(x-3)}{(5-2x)^2(7-2x)^2} < 0$ for $x \geq 4$. Further, we have:

$$\begin{aligned} \textstyle\int_4^{\infty} \frac{1}{(2x-5)(2x-7)}\,dx \;&=\; \textstyle\int_4^{\infty} -\frac{\frac{1}{2}}{2x-5} + \frac{\frac{1}{2}}{2x-7}\;dx \\ &=\; \textstyle\lim_{n\to\infty} \frac{1}{4} \ln\!\left(\frac{2x-7}{2x-5}\right)\Big|_{x=4}^{n} \;=\; 0 - \ln(\frac{1}{3}) \;<\; \infty. \end{aligned}$$

since $\lim_{n\to\infty} \ln\!\left(\frac{2n-7}{2n-5}\right) = \ln(1) = 0$.

Slightly generalizing the reasoning of the convergent case of the Integral Test above, we find that $\sum\limits_{n=5}^{\infty} a_n \leq \int_4^{\infty} f(x)\,dx$. Thus, we have:

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \;&=\; a_1 + a_2 + a_3 + a_4 + \sum_{n=5}^{\infty} a_n \\ &<\; a_1 + a_2 + a_3 + a_4 + \textstyle\int_4^{\infty} f(x)\,dx \;<\; \infty. \end{aligned}$$

†In fact, Euler computed this limit as $\frac{\pi^2}{6} \approx 1.64$. Look up the Basel Problem.