Review. In §11.10, we saw how Taylor series compute any reasonable function $f(x)$ as a kind of "infinite polynomial" near a center point $x=a$:

$$f(x) \;=\; \sum_{n=0}^{\infty} \tfrac{f^{(n)}(a)}{n!}(x{-}a)^n \;=\; f(a) + f'(a)(x{-}a) + \tfrac{f''(a)}{2!}(x{-}a)^2 + \tfrac{f'''(a)}{3!}(x{-}a)^3 + \cdots,$$

where $n! = n(n{-}1)\cdots(2)(1)$ with $0!=1$. The constant coefficients $c_n = \frac{f^{(n)}(a)}{n!}$ involve the nth derivatives $f^{(n)}(x)$, but use their values *only* at the center point $x=a$: if the formula to compute $f(x)$ required that we know $f(x)$, it would be useless.

The first two terms $f(x) \approx f(a) + f'(a)(x{-}a)$ make the linear approximation, while the degree N Taylor polynomial $T_N(x) = \sum_{n=1}^{N} \frac{f^{(n)}(a)}{n!}(x{-}a)^n$ gives a better and better approximation of $f(x)$ as we take more terms, provided x is in the interval of convergence.* This is how calculators can accurately compute complicated functions using only the four arithmetic operations.

In this section, we consider only Maclaurin series $f(x) = \sum\limits_{n=0}^{\infty} c_n x^n$, centered at $x=0$.

Binomial series. We have seen several functions which have simple series because their nth derivatives are easy to compute, at least at $x=0$. One of the most useful of these is the *binomial series*, the Maclaurin series for the function $f(x) = (1{+}x)^p$, the pth power of a binomial (the sum of two terms). The coefficients of the series $(1{+}x)^p = \sum_{n=0}^{\infty} c_n x^n$ are called *binomial coefficients*, and they have a special symbol $c_n = \binom{p}{n}$.† By definition:

$$f(x) \;=\; (1{+}x)^p \;=\; \sum_{n=0}^{\infty} \binom{p}{n} x^n \;=\; \binom{p}{0} + \binom{p}{1}x + \binom{p}{2}x^2 + \cdots.$$

We compute these by the usual formula: $\binom{p}{n} = c_n = \frac{f^{(n)}(0)}{n!}$. The nth derivative is:

$$f^{(n)}(x) = p(p{-}1)\cdots(p{-}n{+}1)\,(1{+}x)^{p-n},$$

so plugging in $x=0$ gives:

$$\binom{p}{n} \;=\; \frac{\overbrace{p(p{-}1)\cdots(p{-}n{+}1)}^{n \text{ factors}}}{n!}.$$

The Ratio Test shows that any binomial series has radius of convergence $|x|<1$, except when p is a whole number.

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*The Lagrange Remainder Formula below bounds the error in the approximation $f(x) \approx T_N(x)$. By the Ratio Test, the series $f(x) = \sum_{n=1}^{\infty} c_n(x{-}a)^n$ will converge if $|x{-}a| < R$, where the radius of convergence is: $R = \lim\limits_{n\to\infty} |c_n/c_{n+1}|$. If R is finite, the open interval of convergence is $x \in (a{-}R, a{+}R)$.

†The symbol $\binom{p}{n}$ is usually read "p choose n" because if p is a whole number, it turns out that $\binom{p}{n}$ is the number of ways, given a set of p objects, to choose a subset of n of them. For example, $\binom{4}{2} = \frac{6\cdot 5}{2\cdot 1} = 6$ counts 6 ways to choose 2 numbers from $\{1,2,3,4\}$, i.e. $\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$.

EXAMPLE: For $p = \frac{1}{2}$ and $f(x) = (1{+}x)^{1/2} = \sqrt{1{+}x}$, we get a series very much like that for $\sqrt{x}$ in §11.10:

$$\begin{aligned}(1{+}x)^{1/2} &= 1 + \tfrac{1}{2}x + \frac{\frac{1}{2}\cdot(-\frac{1}{2})}{2!}\,x^2 + \frac{\frac{1}{2}\cdot(-\frac{1}{2})\cdot(-\frac{3}{2})}{3!}\,x^3 + \frac{\frac{1}{2}\cdot(-\frac{1}{2})\cdot(-\frac{3}{2})\cdot(-\frac{5}{2})}{4!}\,x^4 + \cdots . \\ &= 1 + \tfrac{1}{2}x - \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{1}{2!}\,x^2 + \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{3!}\,x^3 - \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tfrac{5}{2}\cdot\tfrac{1}{4!}\,x^4 + \cdots . \\ &= 1 + \tfrac{1}{2}x + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{(2n{-}3)!!}{2^n n!}x^n \quad \text{for } |x|<1,\end{aligned}$$

where we use the odd-factorial notation: $(2n{-}3)!! = 1{\cdot}3{\cdot}5{\cdot}7\cdots(2n{-}3)$.

EXAMPLE: For $p = -1$ we get a geometric series with ratio $r = -x$:

$$(1{+}x)^{-1} \;=\; 1 + (-1)x + \tfrac{(-1)(-2)}{2!}x^2 + \tfrac{(-1)(-2)(-3)}{3!}x^3 + \cdots \;=\; 1 - x + x^2 - x^3 - \cdots .$$

For $p = -2$ (and similarly for any negative integer), the binomial series also simplifies:

$$(1{+}x)^{-2} \;=\; 1 + (-2)x + \tfrac{(-2)(-3)}{2!}x^2 + \tfrac{(-2)(-3)(-4)}{3!}x^3 + \cdots \;=\; 1 - 2x + 3x^2 - 4x^3 - \cdots ,$$

which we obtained in §11.9 as the derivative of $(1{+}x)^{-1}$.

Whole number powers. If p is a positive integer, the function $(1{+}x)^p$ mulitplies out to a polynomial, so it has a *finite* series with highest non-zero term x^p: that is, all higher terms have coefficient zero, and can be dropped. For example, taking $p = 5$:

$$\begin{aligned}(1{+}x)^5 \;&=\; \textstyle\sum_{n=0}^{\infty}\binom{p}{n}x^n \;=\; 1 + \tfrac{5}{1!}x + \tfrac{5\cdot4}{2!}x^2 + \tfrac{5\cdot4\cdot3}{3!}x^3 + \tfrac{5\cdot4\cdot3\cdot2}{4!}x^4 + \tfrac{5\cdot4\cdot3\cdot2\cdot1}{5!}x^5 + 0x^6 + \cdots \\ &=\; 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 .\end{aligned}$$

Indeed, taking $x = \frac{b}{a}$ and clearing denominators gives a general algebraic formula analogous to $(a+b)^2 = a^2 + 2ab + b^2$:

$$(a+b)^5 \;=\; a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 .$$

Of course, we could also obtain this by successively multiplying out powers of $(a+b)$:

$$\begin{aligned}(a+b)^0 &= 1 \\ (a+b)^1 &= a+b \\ (a+b)^2 &= a^2+2ab+b^2 \\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\ (a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\ (a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5\end{aligned}$$

In the pth row, the coefficients are:

$$\tbinom{p}{0} = 1 \qquad \tbinom{p}{1} = p \qquad \tbinom{p}{2} = \tfrac{1}{2}p(p{-}1) \quad \cdots \quad \tbinom{p}{p-1} = p \qquad \tbinom{p}{p} = 1 .$$

Because each row is obtained by multiplying the previous by $(a+b)$, each coefficient is the sum of the two immediately above it to the left and right, for example $10 = 4 + 6$. The array of whole-number coefficients, continuing downward infinitely, is called *Pascal's Triangle*; it occurs in many problems in algebra and probability.

Modifications of series. Once we know a series formula

$$f(x) \;=\; c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \;=\; \sum_{n=0}^{\infty} c_n x^n$$

for some explicit coefficients c_n, we can manipulate it to get new series formulas for similar functions. Let k be a fixed positive integer, and q a constant.

- $qx^k f(x) \;=\; qc_0 x^k + qc_1 x^{k+1} + qc_2 x^{k+2} + qc_3 x^{k+3} + \cdots \;=\; \sum_{n=k}^{\infty} qc_{n-k} x^n.$
- $f(qx^k) \;=\; c_0 + c_1 q x^k + c_2 q^2 x^{2k} + c_3 q^3 x^{3k} + \cdots \;=\; \sum_{n=0}^{\infty} c_k q^k x^{kn}.$
- $\displaystyle\int f(x)\,dx \;=\; c_0 x + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{3} + c_3 \frac{x^4}{4} + \cdots \;=\; \sum_{n=0}^{\infty} c_n \frac{x^{n+1}}{n+1}.$

(We already saw the last modification in §11.9.)

EXAMPLE: Find the explicit Maclaurin series of $f(x) = \frac{x^2+1}{\sqrt[3]{x^2-1}}$. We manipulate this function to write it in terms of the known binomial series $(1+x)^{-1/3} = \sum_{n=0}^{\infty} \binom{-1/3}{n} x^n$.

$$\begin{aligned}
\frac{x^2+1}{\sqrt[3]{x^2-1}} &= -x^2(1-x^2)^{-1/3} \;-\; (1-x^2)^{-1/3} \\
&= -x^2 \sum_{n=0}^{\infty} \binom{-1/3}{n} (-x^2)^n \;-\; \sum_{n=0}^{\infty} \binom{-1/3}{n} (-x^2)^n \\
&= \sum_{n=0}^{\infty} (-1)^{n+1} \binom{-1/3}{n} x^{2n+2} \;-\; \sum_{n=0}^{\infty} (-1)^n \binom{-1/3}{n} x^{2n} \\
&= \sum_{n=1}^{\infty} (-1)^n \binom{-1/3}{n-1} x^{2n} \;-\; \sum_{n=0}^{\infty} (-1)^n \binom{-1/3}{n} x^{2n} \\
&= -1 + \sum_{n=1}^{\infty} (-1)^n \left[\binom{-1/3}{n-1} - \binom{-1/3}{n} \right] x^{2n}.
\end{aligned}$$

A tricky point is the index shift from $n = 0$ to $n = 1$ in the left summation: you can see in dot-dot-dot notation that the two ways of indexing produce the same terms:

$$\begin{aligned}
\sum_{n=0}^{\infty} (-1)^{n+1} \binom{-1/3}{n} x^{2n+2} &= (-1)^1 \binom{-1/3}{0} x^2 + (-1)^2 \binom{-1/3}{1} x^4 + (-1)^3 \binom{-1/3}{2} x^6 + \cdots \\
&= \sum_{n=1}^{\infty} (-1)^n \binom{-1/3}{n-1} x^{2n}.
\end{aligned}$$

Also, in the term $-\sum_{n=0}^{\infty} (-1)^n \binom{-1/3}{n} x^{2n}$ on the fourth line of the computation, we broke off the $n = 0$ term $-(-1)^0 \binom{-1/3}{0} = -1$, leaving a summation starting at $n = 1$.

EXAMPLE: Find the explicit Maclaurin series of $f(x) = x\sin(x^2) - x + x^3$. We write this in terms of the known trig series $\sin(x) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1}$.

$$\begin{aligned} x\sin(x^2) - x + x^3 &= -x + x^3 + x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,(x^2)^{2n+1} \\ &= -x + x^3 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{4n+3} \\ &= -x + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{4n+3}. \end{aligned}$$

Here $-x^3$ canceled the $n=0$ term $\frac{(-1)^0}{(2(0)+1)!}x^{4(0)+3} = x^3$, leaving $n \geq 1$.

EXAMPLE: Find the explicit Maclaurin series of the indefinite integral $\int e^{-x^2}\,dx$, the *Gaussian error function.* This integral cannot be computed algebraically, though of course we could numerically approximate $F(x) = \int_0^x e^{-t^2}\,dt$ for a given x using Riemann sums. An alternative is the Taylor series, whose finite sums give approximations to the integral function $F(x)$ for all x:

$$\int e^{-x^2}\,dx \;=\; \int \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\,dx \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n}\,dx \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}.$$

Bounding the remainder to determine accuracy. For a function with Taylor series $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$, we define the *remainder term* as the difference between a function and its Taylor polynomial approximation:

$$R_N(x) \;=\; f(x) - T_N(x) \;=\; \sum_{n=N+1}^{\infty} c_n(x-a)^n.$$

Thus $f(x) = T_N(x) + R_N(x)$, so that $R_N(x)$ is the error in approximating $f(x) \approx T_N(x)$.

> *Lagrange Remainder Formula:* For any Taylor polynomial approximation $f(x) = T_N(x) + R_N(x)$, the remainder term is equal to:
>
> $$R_N(x) \;=\; \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$$
>
> for some point c between a and x.

This allows an *a priori* estimate of the error, provided we can find an upper bound for the derivative: if $|f^{(N+1)}(t)| \leq M$ for all $t \in [a,x]$ or $[x,a]$, then this holds for the particular value $t = c$ in the Remainder Formula, and:

$$|R_N(x)| \;\leq\; \max_{t\in[a,x]} \left| \frac{f^{(N+1)}(t)}{(N+1)!}(x-a)^{N+1} \right| \;\leq\; \frac{M}{(N+1)!}\,|x-a|^{N+1}.$$

This generalizes the error estimate for the linear approximation (Calculus I §2.9 end and §3.2 end). Note the error expression $\frac{1}{(N+1)!}f^{(N+1)}(c)\,(x-a)^{N+1}$ is almost the same as the next series term $\frac{1}{(N+1)!}f^{(N+1)}(a)\,(x-a)^{N+1}$: the only difference is taking $f^{(N+1)}$ at a middle point $c \in (a,x)$ instead of the endpoint a. We give proofs below.

EXAMPLE: In §11.10 we computed $\sin(\frac{\pi}{18}) = T_3(\frac{\pi}{18}) + R_3(\frac{\pi}{18}) = 0.1736468 + R_3(\frac{\pi}{18})$, centered at $a = 0$. We have the upper bound:

$$|f^{(N+1)}(t)| \;=\; |\sin^{(4)}(t)| \;=\; |\sin(t)| \;\leq\; M \;=\; 1 \quad \text{for} \quad t \in [0, \tfrac{\pi}{18}]\,.$$

Thus, the error term is at most:

$$\left|R_3(\tfrac{\pi}{18})\right| \;\leq\; \tfrac{M}{(N+1)!}\,|x-a|^{N+1} \;=\; \tfrac{1}{4!}(\tfrac{\pi}{18})^4 \;\approx\; 4\times 10^{-5}\,.$$

Approximation to n decimal places means with error smaller than 0.5×10^{-n}, so our approximation ($\pm 0.4\times 10^{-4}$) is accurate to at least 4 places (though actually 5 places).

EXAMPLE: In §11.10 we computed $\sqrt{2} = T_4(2) + R_4(2) = 1.4142143 + R_4(2)$, centered at $a = \frac{9}{4}$. We have the upper bound:

$$|f^{(N+1)}(t)| \;=\; |\tfrac{d^5}{dt^5}(t^{1/2})| \;=\; \tfrac{1\cdot1\cdot3\cdot5\cdot7}{2\cdot2\cdot2\cdot2\cdot2}\,t^{-9/2} \;\leq\; \tfrac{1\cdot1\cdot3\cdot5\cdot7}{2\cdot2\cdot2\cdot2\cdot2}\,2^{-9/2} \approx 0.15 \;<\; M \;=\; 0.2$$

for $t \in [2, \frac{9}{4}]$: we plug in the left endpoint $t = 2$ since $t^{-9/2}$ is a decreasing function. Thus, the error term is at most:

$$|R_4(2)| \;\leq\; \tfrac{M}{(N+1)!}\,|x-a|^{N+1} \;=\; \tfrac{0.2}{5!}\,\left|2-\tfrac{9}{4}\right|^5 \;\approx\; 2\times 10^{-6}\,.$$

Our approximation is accurate to at least 5 decimal places.

EXAMPLE: Smooth but not analytic. The function $f(x) = e^{-1/x^2}$ is undefined at $x = 0$ because $1/x^2$ is undefined. However, we can easily check that $\lim_{x\to 0} f(x) = 0$, so $x = 0$ is a removable discontinuity (§1.8), and we can just define $f(0) = 0$. The resulting function is continuous and has all derivatives zero: $f^{(n)}(0) = \lim_{x\to 0} f^{(n)}(x) = 0$.

Does $f(x)$ have a Taylor series centered at $x = 0$? If so, it would have coefficients $c_n = \frac{1}{n!}f^{(n)}(0) = 0$, producing the trivial Taylor series $f(x) \overset{??}{=} 0 + 0x + 0x^2 + \cdots$, which is clearly nonsense. We say that this function is *smooth* at $x = 0$ since it has derivatives of all orders, but not *analytic* since it has no convergent Taylor series.

The Lagrange Remainder Formula still holds for the trivial series, but it does not provide a useful ceiling for $R_N(x)$: for a fixed small $|x| > 0$ and $N \to \infty$, the factor $\frac{|x|^{N+1}}{(N+1)!}$ gets very small, but the numerator $f^{(N+1)}(c)$ gets very large, and instead of shrinking, $R_N(x)$ stays constant as $N \to \infty$. The problem is that very near $x = 0$, the derivative $|f^{(N+1)}(x)|$ has a very steep canyon with bottom $f^{(N+1)}(0) = 0$ between two very tall spikes, allowing large $f^{(N+1)}(c)$.

CHALLENGE PROBLEM: Prove the described behavior of $f^{(N)}(x)$ near $x = 0$, and show that the spikes are at approximately $x = \pm\sqrt{2/N}$. Hint: Consider the substitution $z = 1/x$, and use the techniques of §6.8.

Proof of Remainder Bound. The First Fundamental Theorem (§4.3) gives $f(x) = f(a) + \int_a^x f'(t)\,dt$. Integrating by parts, $\int_a^x u\,dv = uv|_{t=a}^{t=x} - \int_a^x v\,du$ with $u = f'(t)$, $du = f''(t)\,dt$, $v = x{-}t$, $dv = -dt$:

$$\begin{aligned} f(x) &= f(a) - \textstyle\int_a^x f'(t)\,(x{-}t)'dt \\ &= f(a) - \big(f'(x)(x{-}x) - f'(a)(x{-}a)\big) + \textstyle\int_a^x f''(t)(x{-}t)\,dt \\ &= f(a) + f'(a)(x{-}a) + \textstyle\int_a^x f''(t)(x{-}t)\,dt, \end{aligned}$$

which means $R_1(x) = \int_a^x f''(t)(x{-}t)\,dt$. Repeating with $u = f''(t)$ and $v = \frac{1}{2}(x{-}t)^2$:

$$f(x) \;=\; f(a) + f'(a)(x{-}a) + \tfrac{1}{2}f''(a)(x{-}a)^2 + \tfrac{1}{2}\textstyle\int_a^x f'''(t)(x{-}t)^2\,dt,$$

so that $R_2(x) = \frac{1}{2}\int_a^x f'''(t)(x{-}t)^2\,dt$. Continuing in this way gives:

$$R_N(x) \;=\; \frac{1}{N!}\int_a^x f^{(N+1)}(t)\,(x{-}t)^N\,dt.$$

Thus $|f^{(N+1)}(t)| < M$ implies the weak bound $|R_N(x)| \le \frac{M}{N!}(x{-}a)^{N+1}$, omitting factor $\frac{1}{N+1}$.

To get the full Lagrange remainder formula and the consequent remainder bound, hold x constant and define the function $r(t) = \frac{1}{N!}f^{(N+1)}(t)\,(x{-}t)^N$, so that $R_N(x) = \int_a^x r(t)\,dt$ by the above computations. Applying the integral form of the Cauchy Mean Value Theorem (see §3.2 & §4.4) to the functions $r(t)$ and $g(t) = (x{-}t)^N$, we find that there exists $c \in (a,x)$ such that $r(c)/g(c) = (\int_a^x r(t)\,dt)/(\int_a^x g(t)\,dt)$, i.e.

$$\frac{\frac{1}{N!}f^{(N+1)}(c)\,(x{-}c)^N}{(x{-}c)^N} \;=\; \frac{R_N(x)}{-\frac{1}{N+1}(x{-}x)^{N+1} + \frac{1}{N+1}(x{-}a)^{N+1}}.$$

Simplifying gives $R_N(x) = \frac{1}{(N+1)!}f^{(N+1)}(c)\,(x{-}a)^{N+1}$ as desired.

Quadratic convergence of Newton's Method. Recall our other main numerical method from Calculus I, §3.8: Newton's Method finds approximate solutions to an equation $g(x) = 0$ by repeatedly solving a linear approximation of $g(x) = 0$ to improve an approximate solution x_n to:

$$x_{n+1} \;=\; x_n - \frac{g(x_n)}{g'(x_n)}.$$

For example, we saw in §11.10 how to approximate $\sqrt{2}$ either by finding a Taylor series for $f(x) = \sqrt{x}$ and plugging in $x = 2$; or by using Newton's Method to solve the equation $g(x) = x^2 - 2 = 0$. (However, Newton's Method cannot compute transcendental functions like $\sin(x)$ or e^x.)

Using the Lagrange Remainder Formula, we can show that each term of a Taylor series adds about a constant number of accurate decimal places to the approximation. Newton's Method is much more powerful: each iteration roughly *doubles* the number of accurate decimal places.

> *Newton's Method error bound:* Suppose for x in some interval, the function $g(x)$ has a root $g(r) = 0$, and for all $x = c_1, c_2$ it obeys $g'(c_2) \neq 0$ and $|g''(c_1)/g'(c_2)| < M$.
>
> Then the errors of approximate root x_n and the improved approximation x_{n+1} obey:
>
> $$|x_{n+1} - r| \;\le\; \tfrac{1}{2}M\,|x_n - r|^2.$$

Proof. The Lagrange Remainder Formula centered at $x = x_n$ says that for some $c \in (x_n, r)$:

$$0 \;=\; g(r) \;=\; g(x_n) + g'(x_n)(r{-}x_n) + \tfrac{1}{2}g''(c)(r{-}x_n)^2,$$

$$\frac{g(x_n)}{g'(x_n)} \;=\; x_n - r - \frac{g''(c)(x_n{-}r)^2}{2g'(x_n)},$$

$$x_{n+1} - r \;=\; \left(x_n - \frac{g(x_n)}{g'(x_n)}\right) - r \;=\; \frac{g''(c)}{2g'(x_n)}(x_n{-}r)^2.$$

Hence the bound $|g''(c)/g'(x_n)| \le M$ implies: $|x_{n+1}{-}r| \le \frac{1}{2}M\,|x_n{-}r|^2$ as desired.