Math 133 # Sequences **Stewart §11.1**

Real functions and sequences. So far, our main objects of study have been functions $f : \mathbb{R} \to \mathbb{R}$, where the inputs and outputs are in the set of real numbers $\mathbb{R} = (-\infty, \infty)$. In this chapter, we introduce a new type of function called a *sequence*:

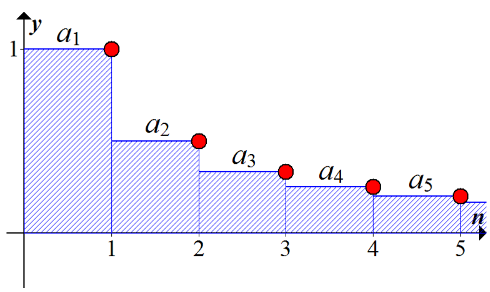
$$a : \{1, 2, 3, \ldots\} \to \mathbb{R}\,,$$

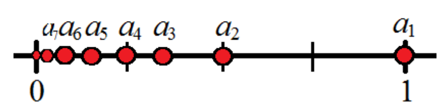
in which the inputs are whole numbers $n = 1, 2, 3, \ldots$, and the outputs are again real numbers, usually written as a_n instead of $a(n)$. The index n can be replaced arbitrarily: a_i for $i = 1, 2, 3, \ldots$ is the same sequence as a_n. Some sequences may begin with a_0.

We can write a sequence either as a formula or as a list of outputs; for example:

$$a_n = \tfrac{1}{n} \qquad \Longleftrightarrow \qquad \{a_n\}_{n=1}^{\infty} = 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots$$

Here $\{a_n\}_{n=1}^{\infty}$ denotes the entire sequence, thought of as an infinite list, and we write the first few values $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, with dot-dot-dot $(\ldots)$ meaning "continue this pattern". We can picture this by plotting the points (n, a_n) in the plane, sometimes with a bar-graph as at left; or by marking only the output values $a_1, a_2, a_3, \ldots$ on a number line as at right:





EXAMPLES:

- $\{a_n\} = 1, -1, 1, -1, \ldots \quad \Longleftrightarrow \quad a_n = \begin{cases} 1 & \text{for } n \text{ odd} \\ -1 & \text{for } n \text{ even} \end{cases} \quad \Longleftrightarrow \quad a_n = (-1)^{n-1}$

- $a_n = \sin\left(\frac{n\pi}{2}\right) \quad \Longleftrightarrow \quad a_n = \begin{cases} 0 & \text{for } n \text{ even} \\ 1 & \text{for } n = 4k+1 \text{ with integer } k \\ -1 & \text{for } n = 4k+3 \text{ with integer } k \end{cases}$

  $\Longleftrightarrow$

n	1	2	3	4	5	6	7	8	$\cdots$
a_n	1	0	-1	0	1	0	-1	0	$\cdots$

- $a_n = 2^n \quad \Longleftrightarrow \quad \{a_n\} = 2, 4, 8, 16, \ldots \quad \Longleftrightarrow \quad a_1 = 2,\; a_n = 2a_{n-1} \text{ for } n \geq 2$

  The last definition is *recursive*, meaning that each value a_n is defined in terms of the previous value a_{n-1}, starting with an initial value $a_1 = 2$,

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- The *Fibonacci sequence* is the most famous recursive sequence: each entry is the sum of the previous two.

$$F_1 = F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \ \text{ for } n \geq 3.$$

n	1	2	3	4	5	6	7	8	$\cdots$
F_n	1	1	2	3	5	8	13	21	$\cdots$

There is no obvious formula for F_n in terms of n, but look up *Binet's formula.*

Convergence. It would be meaningless to take the limit of a sequence a_n as $n \to c$, since a whole number n cannot gradually approach a finite value c. However, we can take the limit as $n \to \infty$.

> *Definition:* We say the sequence $\{a_n\}_{n=1}^{\infty}$ *converges* to the number L, denoted $\lim_{n\to\infty} a_n = L$, whenever a_n gets as close as desired to L, provided n is large enough. Specifically, for any error tolerance $\epsilon > 0$, there is some lower bound N such that $n > N$ forces $L - \epsilon < a_n < L + \epsilon$; or equivalently:
>
> $$n > N \quad \Longrightarrow \quad |a_n - L| < \epsilon\,.$$
>
> If the limit does not exist, we say the sequence *diverges.*

This just repeats the error-control definition for $\lim_{x\to\infty} f(x) = L$ from §1.7, and we have a similar definition for divergence to infinity, $\lim_{n\to\infty} a_n = \infty$ or $-\infty$. In the pictures above, we can see the convergence of $a_n = \frac{1}{n}$ to $L = 0$: in the graph, we see the points (n, a_n) approach the horizontal asymptote $y = L$; on the number line, we see the a_n points march right up to the limit value L.

EXAMPLE: Prove that: $\lim_{n\to\infty} 2 + \frac{(-1)^n}{2n} = 2$. Given the acceptable error tolerance $\epsilon > 0$, we work backward from the desired inequality:

$$2 - \epsilon < 2 + \frac{(-1)^n}{2n} < 2 + \epsilon \quad \Longleftrightarrow \quad -\epsilon < \frac{(-1)^n}{2n} < \epsilon \quad \Longleftrightarrow \quad \left|\frac{(-1)^n}{2n}\right| < \epsilon \quad \Longleftrightarrow \quad n > \frac{1}{2\epsilon}$$

For example, if we want $|a_n - 2| < \epsilon = \frac{1}{100}$, we take $n > \frac{1}{2\epsilon} = \frac{1}{2/100} = 50$.

Limit Laws. We do not usually perform error-control analysis to work with limits of sequences a_n, but rather rely on our previous knowledge of limits of functions $f(x)$:

> *Sequence Comparison Theorem:* If $f(x)$ is a function with $a_n = f(n)$ for all n, then $\lim\limits_{n\to\infty} a_n = \lim\limits_{x\to\infty} f(x)$, when the right-hand limit exists or is $\pm\infty$.

EXAMPLE: Compute $\lim_{n\to\infty} \frac{n^2+n}{2n^2-3}$. Here $a_n = \frac{n^2+n}{2n^2-3}$ for $n = 1, 2, 3, \ldots$ is the sequence version of $f(x) = \frac{x^2+x}{2x^2-3}$ for real numbers x, and we have techniques to deal with limits of $f(x)$. Here, we can use L'Hôpital's Rule:

$$\lim_{n\to\infty} \frac{n^2+n}{2n^2-3} \;=\; \lim_{x\to\infty} \frac{x^2+x}{2x^2-3} \;\overset{\text{Hop}}{=}\; \lim_{x\to\infty} \frac{2x+1}{4x} \;\overset{\text{Hop}}{=}\; \lim_{x\to\infty} \frac{2}{4} \;=\; \frac{1}{2}\,.$$

We cannot use L'Hôpital's Rule directly on a_n because we cannot take the derivative of a sequence: it is not a curve with a slope at each point.

An alternative way of handling limits of sequences is to repeat the kind of analysis we did with functions: combine Basic Limits using Limit Laws (§1.6). We have:

- *Basic Limits:* $\lim\limits_{n\to\infty} c = c$ and $\lim\limits_{n\to\infty} n = \infty$.
- *Sum Law:* $\lim\limits_{n\to\infty} a_n + b_n = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$.
- *Product Law:* $\lim\limits_{n\to\infty} a_n b_n = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$.
- *Quotient Law:* $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$.

The above Laws are valid provided the right-side expressions make sense: for example, in the Quotient Law we must assume that a_n and b_n converge, and $\lim\limits_{n\to\infty} b_n \neq 0$. Furthermore, the Laws are valid when the right-side limits are infinite, provided we use the Infinity Rules:

$$\infty + \infty = \infty \qquad \infty \cdot \infty = \infty \qquad c\cdot\infty = \begin{cases} \infty & \text{if } c > 0 \\ -\infty & \text{if } c < 0 \end{cases} \qquad \frac{1}{\pm\infty} = 0\,.$$

EXAMPLE: We can re-do the sequence in the previous example as follows:

$$\lim_{n\to\infty} \frac{n^2+n}{2n^2-3} = \lim_{n\to\infty} \frac{n^2+n}{2n^2-3} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2-\frac{3}{n^2}}$$

Applying the Limit Laws and Infinity Rules, this becomes:

$$\frac{1+\lim\limits_{n\to\infty}\frac{1}{n}}{2-3\Big(\lim\limits_{n\to\infty}\frac{1}{n}\Big)^2} = \frac{1+\frac{1}{\infty}}{2-3\big(\frac{1}{\infty}\big)^2} = \frac{1+0}{2-3(0^2)} = \frac{1}{2}\,.$$

Limit Theorems. We have two more results which parallel those for limits of $f(x)$:

> *Squeeze Theorem:* If $a_n \le b_n \le c_n$ for all n, and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.

EXAMPLE: Rigorously evaluate the limit of $b_n = \frac{2+\sin(n^2)}{n}$. Note that the sequence $q_n = \sin(n^2)$ diverges, oscillating unpredictably between $-1 \le \sin(n^2) \le 1$. However, we have bounds:

$$a_n = \frac{1}{n} = \frac{2-1}{n} \le \frac{2+\sin(n^2)}{n} \le \frac{2+1}{n} = \frac{3}{n} = c_n\,.$$

Since the upper and lower bounds both approach the limit $L = 0$, so does the middle sequence: $\lim\limits_{n\to\infty} \frac{2+\sin(n^2)}{n} = 0$.

Continuity Theorem: If $g(x)$ is continuous, i.e. $\lim_{x\to c} g(x) = g(c)$ for all c, then:

$$\lim_{n\to\infty} g(a_n) \;=\; g\Big(\lim_{n\to\infty} a_n\Big)\,.$$

EXAMPLE: Find $\lim_{n\to\infty} n^{1/n}$: that is, does the sequence $1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \ldots$ approach a finite value? As always with exponentiation, we rewrite in terms of the natural exponential $\exp(x) = e^x$, which is a continuous function:

$$\lim_{n\to\infty} n^{1/n} \;=\; \lim_{n\to\infty} e^{\ln(n)/n} \;=\; \lim_{n\to\infty} \exp\left(\frac{\ln(n)}{n}\right) \;=\; \exp\left(\lim_{n\to\infty} \frac{\ln(n)}{n}\right).$$

Now we can evaluate the inside limit by L'Hôpital:

$$\lim_{n\to\infty} \frac{\ln(n)}{n} \;=\; \lim_{x\to\infty} \frac{\ln(x)}{x} \;=\; \lim_{x\to\infty} \frac{1/x}{1} \;=\; 0.$$

Hence $\lim_{n\to\infty} n^{1/n} = e^0 = 1$. Check this by computing values of $n^{1/n}$ on your calculator.

Continuous compounding. Here is a surprising example from financial theory. Suppose a bank account pays an annual interest rate of r: for example, $r = 0.04 = 4\%$ means that after a year, each dollar becomes $1 + r = 1.04$ dollars.

Now suppose half the interest is paid after half a year, giving $1 + \frac{r}{2}$ dollars, and in the second half-year, the previous interest also earns interest (i.e. compound interest). At the end of the year, each dollar becomes $(1 + \frac{r}{2})(1 + \frac{r}{2}) = (1 + \frac{r}{2})^2$ dollars. If the interest is paid three times a year, compound interest gives $(1 + \frac{r}{3})^3$ dollars; and if interest is paid n times a year, it gives $(1 + \frac{r}{n})^n$ dollars.

Now imagine if interest were paid every hour, or every second, etc., approaching a system of compounding continuously at every instant. Would this produce an unbounded amount of money, or tend to a limit? Let's see!

$$\Big(1 + \frac{r}{n}\Big)^n \;=\; \exp\Big(\ln\Big(1 + \frac{r}{n}\Big)\, n\Big)$$

Now L'Hôpital gives:

$$\lim_{x\to\infty} \ln\Big(1 + \frac{r}{x}\Big)\; x \;=\; \lim_{x\to\infty} \frac{\ln\big(1 + rx^{-1}\big)}{x^{-1}} \;\overset{\text{Hop}}{=}\; \lim_{x\to\infty} \frac{\frac{1}{1+rx^{-1}}\,(-rx^{-2})}{-x^{-2}}$$

$$=\; \lim_{x\to\infty} \frac{rx}{x+r} \;\overset{\text{Hop}}{=}\; \lim_{x\to\infty} \frac{r}{1} \;=\; r\,.$$

Therefore:

$$\lim_{n\to\infty} \Big(1 + \frac{r}{n}\Big)^n \;=\; \exp(r) \;=\; e^r\,.$$

Thus, an interest rate of r produces an annual yield of e^r under continuous compounding. No intervals of compounding will produce more than this. Once again, the natural exponential intrudes even though the original question had nothing to do with it.