Slope in polar coordinates. We have seen that round, turny shapes are more simply described by polar $r\theta$-equations than rectangular $xy$-equations. In this section, we use polar equations to compute geometric information.

Thus, we consider a polar curve $r = f(\theta)$ for $\theta \in [a, b]$. We split the interval $\theta \in [a, b]$ into a large number $n$ of increments, each of length $\Delta \theta = \frac{b-a}{n}$, with sample points $\theta_1, \ldots, \theta_n$. Here is a typical increment of the curve over $\theta \in [\theta_i, \theta_{i+1}]$, showing the corresponding increments in the coordinates:

Our first problem is to find the slope of this curve at a given $\theta$. It is not the derivative $f'(\theta) = \frac{dr}{d\theta}$, which is the rate of change of the radius with respect to the angle. Rather, the slope is the rate of change of $y = r \sin(\theta) = f(\theta) \sin(\theta)$ with respect to $x = r \cos(\theta) = f(\theta) \cos(\theta)$. That is:

\[
(slope \ at \ \theta) = \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{(f(\theta) \sin(\theta))'}{(f(\theta) \cos(\theta))'} = \frac{f'(\theta) \sin(\theta) + f(\theta) \cos(\theta)}{f'(\theta) \cos(\theta) - f(\theta) \sin(\theta)}.
\]

Area in polar coordinates. Next, we assume $r = f(\theta) \geq 0$ for $\theta \in [a, b]$ to avoid complications with negative radius, and we consider the region inside the curve, defined by $0 \leq r \leq f(\theta)$ for $\theta \in [a, b]$. Again we apply Slice Analysis (§5.2), splitting the area $A$ of this region into $n$ thin wedges $\Delta A_i$ corresponding to $[\theta_i, \theta_{i+1}]$: 
We must compute the wedge area $\Delta A_i$. Since $\Delta \theta$ is tiny, the small curve segments are very close to straight lines, and $\Delta A_i$ is a very thin triangle. Neglecting the small piece with radius larger that $r_i$, the slice $\Delta A_i$ is approximately an isosceles triangle with height $r_i$ and base $r_i \Delta \theta$.* Thus:

$$\Delta A_i \approx \frac{1}{2} \text{(base)} \times \text{(height)} \approx \frac{1}{2} (r_i \Delta \theta) r_i = \frac{1}{2} r_i^2 \Delta \theta.$$ 

Therefore the total area is:

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta A_i = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{2} r_i^2 \Delta \theta = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{2} f(\theta_i)^2 \Delta \theta = \int_{a}^{b} \frac{1}{2} f(\theta)^2 \, d\theta.$$ 

**Arclength in polar coordinates.** Finally, we compute the length of the curve $r = f(\theta)$ for $\theta \in [a, b]$. The length $L$ is a sum of $n$ increments $\Delta L_i$:

Each increment $\Delta L_i$ is approximately a straight line segment. Next to it is the radial segment $\Delta r$ and the tiny circular arc with length $r_i \Delta \theta$, which is also approximately a

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*On a unit circle, an arc of $\theta$ radians has length $\theta$, which is the definition of radian measure. On a circle of radius $r$, and arc of $\theta$ radians has length $r \theta$. 
straight line. We get an approximate right triangle with hypotenuse $\Delta L_i$ and legs $r_i \Delta \theta$ and $\Delta r$, so the Pythagorean Theorem gives:

$$
\Delta L_i \approx \sqrt{(r_i \Delta \theta)^2 + (\Delta r)^2} = \sqrt{\frac{(r_i \Delta \theta)^2 + (\Delta r)^2}{(\Delta \theta)^2}} \Delta \theta = \sqrt{\frac{r_i^2}{(\Delta \theta)^2} + (\frac{\Delta r}{\Delta \theta})^2} \Delta \theta.
$$

Therefore the total arc length is:

$$
L = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta L_i = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{r_i^2 + (\frac{\Delta r}{\Delta \theta})^2} \Delta \theta
$$

$$
= \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{f(\theta_i)^2 + (\frac{\Delta f(\theta_i)}{\Delta \theta})^2} \Delta \theta = \int_{a}^{b} \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta.
$$

Example: exponential spiral. Consider the polar curve:

$$
r = f(\theta) = e^{\theta/2\pi},
$$
called an exponential spiral, logarithmic spiral, or or snail-shell:

![Exponential Spiral](image)

It winds infinitely toward the center with each turn having a radius $e^{-1}$ times the previous one.

What is the length of this curve, from the point $(r, \theta) = (1, 0)$ all the way to the center, that is, for $\theta \in (-\infty, 0]$? The arclength formula gives:

$$
L = \int_{-\infty}^{0} \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta = \int_{-\infty}^{0} \sqrt{2} e^{\theta/2\pi} d\theta
$$

$$
= 2\sqrt{2\pi} e^{\theta/2\pi}_{\theta=-\infty}^{\theta=0} = 2\sqrt{2\pi} e^{0/2\pi} - \lim_{N \to \infty} 2\sqrt{2\pi} e^{-N} = 2\sqrt{2\pi} \approx 8.9.
$$

Next, consider the shaded region enclosed by the same section of the curve, along with the dotted segment $\theta = 0$, $e^{-1} \leq r \leq 1$. The outermost turn of the curve, $\theta \in [-2\pi, 0]$,
sweeps out wedges which fill this whole region, so this interval defines the correct bounds for integration:

\[ A = \int_{-2\pi}^{0} \frac{1}{2} f(\theta)^2 \, d\theta = \int_{-2\pi}^{0} \frac{1}{2} e^{\theta/\pi} \, d\theta = \frac{\pi}{2} e^{\theta/\pi} \bigg|_{\theta=-2\pi}^{\theta=0} = \frac{\pi}{2} \left(1-e^{-2}\right) \approx 1.4. \]

**Areas of intersections.** Consider the polar curve \( r = f(\theta) = 1 - \cos(\theta) \). To picture the function \( f \), we draw its rectangular graph (end of §10.3):

The polar graph is a cardioid (heart-shape), which we draw along with the circle \( r = \frac{1}{2} \).

**PROBLEM:** Find the area of the crescent-shaped region which is inside the cardioid and outside the circle.

We must first determine the intersection points of the two curves, where:

\[ r = 1 - \cos(\theta) = \frac{1}{2} \implies \cos(\theta) = \frac{1}{2} \implies \theta = \pm \frac{\pi}{3} + 2n\pi, \]

where \( n \) is any integer. Since the whole cardioid is traced by \( \theta \in [0, 2\pi] \), we can take all intersection points in this range: \( \theta = \frac{\pi}{3} \) and \( \theta = -\frac{\pi}{3} + 2\pi = \frac{5\pi}{3} \). Now we take the area inside the cardioid \( r = f(\theta) = 1 - \cos(\theta) \), minus the area inside the circle \( r = g(\theta) = \frac{1}{2} \):

\[
A = \int_{a}^{b} \frac{1}{2} f(\theta)^2 - \frac{1}{2} g(\theta)^2 \, d\theta = \int_{\pi/3}^{5\pi/3} \frac{1}{2} \left(1 - \cos(\theta)\right)^2 - \frac{1}{2} \left(\frac{1}{2}\right)^2 \, d\theta \\
= \left[ \frac{5}{8} \theta - \sin(\theta) + \frac{1}{8} \sin(2\theta) \right]_{\theta=\pi/3}^{\theta=5\pi/3} = \frac{7}{8} \sqrt{3} + \frac{5\pi}{6} \approx 4.1.
\]