Tangents of a parametric curve. We have learned how to write a curve parametrically, as the path of a particle whose position at time $t$ is given by two coordinate functions $(x(t), y(t))$ over a time interval $t \in [a, b]$.

Considering the curve as a track on which the particle runs, the tangent line at a point $(x(c), y(c))$ is the path the particle would take if it were suddenly released from the track at time $t = c$, keeping a constant velocity from that moment. The velocity at $t = c$ has horizontal and vertical components $(x'(c), y'(c))$, giving the parametric line:

$$(x(c) + x'(c)t, y(c) + y'(c)t).$$

The components are the linear approximations of $x(t)$ and $y(t)$ near $t = c$, which is appropriate since the tangent is the line which best approximates the curve near the point.

We can convert this parametric line into an $xy$-equation as in §10.1. The slope is the horizontal over the vertical velocity: $m = \frac{y'(c)}{x'(c)}$, and we know the line passes through $(x(c), y(c))$, so we have the point-slope equation:

$$y = \frac{y'(c)}{x'(c)}(x-x(c)) + y(c).$$

Here $(x, y)$ is a general point of the line, but $x(c), y(c), x'(c), y'(c)$ are constants computed from the coordinate functions of the original curve.

To further explain this, we imagine the original curve as the graph of a function $y = f(x)$, meaning $y(t) = f(x(t))$ for all $t$. The Chain Rule gives:

$$y'(t) = f'(x(t)) \cdot x'(t) \iff \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

At time $t = c$ and $x = x(c)$, this gives our previous slope formula:

$$f'(x(c)) = \frac{y'(c)}{x'(c)} \iff \frac{dy}{dx} = \frac{dy}{dt} \frac{dx}{dt}.$$

Tangents of a circle. We find the tangent line to $(x(t), y(t)) = (2 \sin(\pi t), 2 \cos(\pi t))$ at the point $(\sqrt{2}, \sqrt{2})$. First, to picture the curve, we note:

- Since the components are $2\sin$ and $2\cos$ of the same quantity, the curve is a circle of radius 2.
- The full circle is traced by $\pi t \in [0, 2\pi]$, i.e. $t \in [0, 2]$.
- The curve starts at $(x(0), y(0)) = (0, 2)$ on the $y$-axis; it moves clockwise, since the $x$-coordinate $2\sin(\pi t)$ increases for small $t \geq 0$. 

To apply our formulas, we need to know the value \( t = c \) at which the curve passes through the given point: \((x(t), y(t)) = (\sqrt{2}, \sqrt{2})\). That is, we must solve the system of equations:

\[
\begin{align*}
2 \sin(\pi t) &= \sqrt{2} \\
2 \cos(\pi t) &= \sqrt{2}
\end{align*}
\]

\[\iff t = \frac{1}{4}.
\]

We can find a simultaneous solution to both equations precisely because the point lies on the curve. We have \((x'(c), y'(c)) = (2\pi \cos(\pi/4), -2\pi \sin(\pi/4)) = (\sqrt{2\pi}, -\sqrt{2\pi})\), so the tangent line is:

\[
(x(c) + x'(c)t, y(c) + y'(c)t) = (\sqrt{2} + \sqrt{2}\pi t, \sqrt{2} - \sqrt{2}\pi t)
\]

\[y = \frac{y'(c)}{x'(c)}(x-x(c)) + y(c) = \frac{\sqrt{2\pi}}{\sqrt{2\pi}}(x-\sqrt{2}) + \sqrt{2} \iff y = -x + 2\sqrt{2}.
\]

Note that each tangent to the circle is perpendicular to the corresponding radius.

**Tangents of a polynomial curve.** Find the tangent to \((x(t), y(t)) = (t^2, t^3 - 3t)\) at the point \((3, 0)\). This is not a familiar curve, so to picture it, we must plot points by plugging in various values of \(t\):
We see that the curve passes twice through the given point $(3, 0)$. Algebraically:

\[
\begin{aligned}
\begin{cases}
    t^2 = 3 \\
    t^3 - 3t = 0 
\end{cases}
\iff t = \sqrt{3} \text{ or } t = -\sqrt{3}.
\end{aligned}
\]

Note that $t^3 - 3t = 0$ by itself has the solutions $t = 0, \pm \sqrt{3}$, but $t = 0$ does not satisfy the first equation $t^2 = 3$: for time $t = 0$, the curve is at $(0, 0)$, not $(3, 0)$.

Now we can easily find the two tangent lines: $(3 + 3\sqrt{3}t, 6t)$ and $(3 - 3\sqrt{3}t, 6t)$.

**Example:** Which points of this curve have horizontal tangents? The tangent is horizontal when the vertical velocity is zero: $(t^3 - 3t)' = 3t^2 - 3 = 0 \iff t = \pm 1$, corresponding to the points $(1, -2)$ and $(1, 2)$.

**Arclength.** After applying derivatives to parametric curves, we now apply integrals, which compute the size or bulk of geometric objects. The most natural measure of the size of a curve is its arclength. We already computed this for graph curves $y = f(x)$ in §8.1, and now we do the more general parametric case.

We follow the general scheme for computing any measure of size of a geometric object from §5.2. We want the arclength $L$ of a parametric curve $(x(t), y(t))$ for $t \in [a, b]$. We cut the curve into $n$ bits determined by $\Delta t$-increments of $t \in [a, b]$.

Because the bit at the sample point $t_i$ is so short, it is well approximated by a straight segment, and we can use the Pythagorean Theorem to compute its length:

\[
\Delta L_i \approx \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{\frac{\Delta x^2}{\Delta t^2}} \Delta t = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \Delta t.
\]

In the limit as $n \to \infty$, we get $\Delta t \to 0$ and $\frac{dx}{\Delta t} \to \frac{dx}{dt} = x'(t_i)$; similarly for $\frac{dy}{\Delta t}$:

\[
L = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta L_i = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \Delta t = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.
\]

In Newton notation:

\[
L = \int_{a}^{b} \sqrt{x'(t)^2 + y'(t)^2} \, dt.
\]

In fact, the integrand is just the total speed of the particle at time $t$, combining the horizontal and vertical speeds.
EXAMPLE: Compute the circumference length of a circle of radius $r$. The standard parametrization is $(x(t), y(t)) = (r \cos(t), r \sin(t))$ for $t \in [0, 2\pi]$, with derivative $(x'(t), y'(t)) = (-r \sin(t), r \cos(t))$, and length:

$$L = \int_{0}^{2\pi} \sqrt{(-r \sin(t))^2 + (r \cos(t))^2} \, dt = \int_{0}^{2\pi} r \sqrt{\sin^2(t) + \cos^2(t)} \, dt$$

$$= \int_{0}^{2\pi} r \, dt = rt \bigg|_{t=0}^{t=2\pi} = 2\pi r .$$

The integral is so easy because the particle travels at constant speed $r$. This was much harder in §8.1, using our previous formula $L = \int_{-r}^{r} \sqrt{1+f'(x)^2} \, dx$, where $f(x) = \sqrt{r^2 - x^2}$.

EXAMPLE: Find the length of one arch of the cycloid from §10.1: $(x(t), y(t)) = (t - \sin(t), 1 - \cos(t))$ for $t \in [0, 2\pi]$. We have $(x'(t), y'(t)) = (1 - \cos(t), \sin(t))$, so:

$$L = \int_{0}^{2\pi} \sqrt{(1-\cos(t))^2 + (\sin(t))^2} \, dt = \int_{0}^{2\pi} \sqrt{1-2\cos(t) + \cos^2(t) + \sin^2(t)} \, dt$$

$$= \int_{0}^{2\pi} \sqrt{2(1-\cos(t))} \, dt = \int_{0}^{2\pi} 2\sin\left(\frac{t}{2}\right) \, dt = 8 .$$

Here we used the identity $\sin\left(\frac{t}{2}\right) = \sqrt{\frac{1-\cos(t)}{2}}$. 