Integral as antiderivative. In §4.1, we were given a velocity function \( v(t) \), and we wanted to determine the corresponding position function \( s(t) \). First, we computed the distance traveled by the object over a given time \( t = a \) to \( t = b \) by adding up (velocity)\( \times \) (time) over many small time increments of length \( \Delta t \):

\[
\text{distance traveled} = s(b) - s(a) = \lim_{\Delta t \to 0} \sum_{i=1}^{n} v(t_i) \Delta t = \int_{a}^{b} v(t) \, dt.
\]

Assuming an initial value \( s(a) = 0 \), meaning the object is at zero-position at time \( t = a \), we find that the position at time \( t = b \) is the same as distance traveled from \( t = a \) to \( t = b \):

\[
s(b) = s(b) - s(a) = \int_{a}^{b} v(t) \, dt.
\]

The choice of letters for quantities is only suggestive, and does not affect the computations. Instead of a fixed interval \([a, b]\), let us change to \([a, x]\) to suggest that the right endpoint \( t = x \) is variable, while the left endpoint \( t = a \) remains fixed.* We get the position function:

\[
s(x) = \int_{a}^{x} v(t) \, dt.
\]

This always computes an antiderivative function for \( v(t) \), even if it is impossible to get an antiderivative algebraically by reversing differentiation formulas.

**First Fundamental Theorem.** We generalize the above to produce an antiderivative for any continuous function \( f(x) \), namely the integral function.

*Theorem:* Let \( f(x) \) be continuous for all \( x \in [a, b] \) and define the function:

\[
I(x) = \int_{a}^{x} f(t) \, dt.
\]

Then \( I'(x) = f(x) \) for \( x \in (a, b) \), and \( I(x) \) is the unique antiderivative of \( f(x) \) with \( I(a) = 0 \).

Before a formal proof, let us see how the Theorem relates to our velocity-to-position argument above. If we assume there is some antiderivative \( F(x) \) with \( F'(x) = f(x) \) and \( F(a) = 0 \), then we could approximate \( F(x) = F(x) - F(a) \) as the sum of increments of (rate)\( \times \) (time) = \( f(t_i) \Delta t \), and the exact value of \( F(x) \) as \( \Delta t \to 0 \) would be the integral \( I(x) \). However, this does not prove that there really does exist such an antiderivative \( F(x) \), only that if it exists, it must be given by the integral function.

**Proof of Theorem.** We do not yet know a derivative formula for the new function \( I(x) = \int_{a}^{x} f(x) \, dx \), so we must compute from the definition: \( I'(x) = \lim_{h \to 0} \frac{I(x+h) - I(x)}{h} \). We have:

\[
\frac{I(x+h) - I(x)}{h} = \frac{1}{h} \left( \int_{a}^{x+h} f(t) \, dt - \int_{a}^{x} f(t) \, dt \right) = \frac{1}{h} \int_{x}^{x+h} f(t) \, dt,
\]

*We use the new variable \( x \) to avoid \( s(t) \equiv \int_{a}^{t} v(t) \, dt \), which would imply nonsense like \( s(2) \equiv \int_{a}^{2} v(2) \, d2 \).

†Again, we must use different letters for the limit of integration \( x \) and the variable of integration \( t \).*
since \( f_a^{x+h} = f_a^x + f_x^{x+h} \) for all \( h \) (even \( h < 0 \)).

Geometrically, we see that if \( h \) is small enough, the region above \([x, x+h]\) is approximately a rectangle with height \( f(x) \) and width \( h \), so \( \int_x^{x+h} f(x) \, dx \approx f(x)h \), and:

\[
I'(x) \approx \frac{I(x+h) - I(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t) \, dt \approx \frac{1}{h} (f(x)h) = f(x),
\]

with approximations turning into equalities as \( h \to 0 \), as claimed by the Theorem.

However, geometric inspection is not enough for a general proof, because any picture only shows a particular case, and is not numerically precise. To control errors, we take the absolute minimum value \( N \) and the absolute maximum value \( M \) of the continuous function \( f(x) \) on \([x, x+h]\), using the Extremal Value Theorem (§3.1).\(^4\) (To indicate that these depend on \( h \), we write \( N_h, M_h \).) Now, \( N_h \leq f(t) \leq M_h \) for \( t \in [x, x+h] \), so by the Bounds Rule for integrals (§4.2) we have:

\[
((x+h) - x)N_h \leq \int_x^{x+h} f(t) \, dt \leq ((x+h) - x)M_h \implies N_h \leq \frac{1}{h} \int_x^{x+h} f(t) \, dt \leq M_h.
\]

As \( h \) gets very small, the interval \([x, x+h]\) gets closer and closer to the single point \( x \), and the absolute minimum and maximum over this tiny interval must approach \( f(x) \) by continuity: that is, \( \lim_{h \to 0} N_h = \lim_{h \to 0} M_h = f(x) \). Also, by the above we have:

\[
N_h \leq \frac{I(x+h) - I(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t) \, dt \leq M_h.
\]

Applying the Squeeze Theorem for limits (§1.6), we find what we wanted:

\[
I'(x) = \lim_{h \to 0} \frac{I(x+h) - I(x)}{h} = \lim_{h \to 0} N_h = \lim_{h \to 0} M_h = f(x),
\]

As for the last part of the conclusion, it is clear that \( I(a) = \int_a^a f(t) \, dt = 0 \), and there is a unique antiderivative with this initial value by the Antiderivative Theorem (§3.9), which is a version of the Uniqueness Theorem (§3.2). Note how we have used almost all of our previous theory in proving this culminating Theorem.

**Derivative of integral functions.** The above Theorem can be stated as a Basic Derivative formula for \( I(x) = \int_a^x f(t) \, dt \), where \( f(t) \) is continuous:

\[
I'(x) = \frac{d}{dx} \left( \int_a^x f(t) \, dt \right) = f(x).
\]

\(^4\)Here we assume \( h > 0 \). The case \( h < 0 \) is the same except for a few sign changes.
Here $a$ is any constant, $x$ is the input variable, and $t$ is a “dummy variable” which only has meaning inside the integral.

For another function $g(x)$, we can take its composition with $I(x)$. Then the above Basic Derivative together with the Chain Rule (§2.5) implies:

$$I(g(x))' = \frac{d}{dx} \left( \int_a^{g(x)} f(t) \, dt \right) = I'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x).$$

**EXAMPLE:** Find the derivative of $F(x) = \int_{x^3}^2 \sin(t) \, dt$. We have:

$$F'(x) = \frac{d}{dx} \left( \int_{x^3}^2 \sin(t) \, dt \right) = \frac{d}{dx} \left( \int_0^{x^3} \sin(t) \, dt - \int_0^2 \sin(t) \, dt \right) = \sin(x^3) \cdot (x^3)' - \sin(2x) \cdot (2x)' = 3x^2 \sin(x^3) - 2 \sin(2x).$$

**EXAMPLE:** Find the derivative of $F(x) = \int_a^b \sin(t) \, dt$. Here $a, b$ are constants, and hence so are $2a, b^3$. In fact, the right hand side does not depend on the variable $x$, and is a constant function with derivative $F'(x) = 0$! This also follows from the Chain Rule, since $\sin(2a) \cdot 2(a)' = 0$ and $\sin(b^3) \cdot (b^3)' = \sin(b^3) \cdot 3b^2 \cdot (b)' = 0$.

**Sketching integral functions.** Since an antiderivative $I(x) = \int_a^x f(t) \, dt$ might be a completely new function for which no elementary formula is possible, it might seem mysterious. However, we can find its values with sufficient accuracy by computing Riemann sums on a spreadsheet, and plot these to get a good idea of the graph.

A cleverer strategy is to use the derivative $I'(x) = f(x)$ for sketching $y = I(x)$, as in §3.3 and §3.5. That is, the slope of $y = I(x)$ is given by the height of $y = f(x)$.

**EXAMPLE:** Graph the function $I(x) = \int_0^x \sin(t^2) \, dt$. The critical points of $I(x)$ are solutions of $I'(x) = 0$ or undefined, i.e. $f(x) = \sin(x^2) = 0$ (defined for all $x$). This happens when $x^2 = 2k\pi$ for any integer $k$, so the critical points are $x = 0, \pm \sqrt{\pi}, \pm \sqrt{2\pi}, \ldots$. Sign chart:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-\sqrt{2\pi}$</th>
<th>$-\sqrt{\pi}$</th>
<th>0</th>
<th>$\sqrt{\pi}$</th>
<th>$\sqrt{2\pi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I'(x)$</td>
<td>$+$</td>
<td>$-$</td>
<td>0</td>
<td>$+$</td>
<td>$+$</td>
</tr>
<tr>
<td>$I(x)$</td>
<td>$\nearrow$ -0.43</td>
<td>$\searrow$ -0.89</td>
<td>$\nearrow$ 0</td>
<td>$\nearrow$ 0.89</td>
<td>$\searrow$ 0.43</td>
</tr>
</tbody>
</table>

For inflection points, we solve $I''(x) = 0$, i.e. $f'(x) = 2x \cos(x^2) = 0$, so $x = 0, \pm \sqrt{\pi}, \pm \sqrt{\frac{3\pi}{2}}, \ldots$. Thus, the general shape of the graph is clear, and we can get specific points $(b, I(b))$ from computing a Riemann sum for $\int_0^b \sin(t^2) \, dt$. 

![Graph of integral function](image-url)
From the 180° rotational symmetry of the graph, it looks like \( I(x) \) is an odd function, \( I(-x) = -I(x) \). This is because \( f(x) = \sin(x^2) \) is an even function, \( f(-x) = f(x) \), so:

\[
I(-b) = \int_{-b}^{-b} \sin(t^2) \, dt = -\int_{b}^{0} \sin(t^2) \, dt
\]

\[
= -(\text{area under } y = \sin(x^2) \text{ above } x \in [-b, 0])
\]

\[
= -(\text{area under } y = \sin(x^2) \text{ above } x \in [0, b])
\]

\[
= -\int_{0}^{b} \sin(t^2) \, dt = -I(b)
\]

**Second Fundamental Theorem.** This is a trick to easily evaluate many integrals, which we already used to find some exact values in §4.1.

**Theorem:** Suppose \( F(x) \) is some known antiderivative with \( F'(x) = f(x) \). Then:

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a).
\]

That is, if \( f(x) \) is the rate of change of \( F(x) \), then the integral \( \int_{a}^{b} f(x) \, dx \) is the total change of \( F(x) \) from \( x = a \) to \( b \).

**Proof.** Since \( F(x) \) is a particular antiderivative of \( f(x) \), the Uniqueness Theorem (§3.9, §3.2) says that the general antiderivative is \( F(x) + C \) for any constant \( C \). But the First Fundamental Theorem says the integral function \( I(x) = \int_{0}^{x} f(t) \, dt \) is also an antiderivative of \( f(x) \), so we must have \( I(x) = F(x) + C \). Since we know the initial condition \( I(a) = \int_{a}^{a} f(t) \, dt = 0 \), we get \( I(a) = F(a) + C = 0 \), and \( C = -F(a) \). Therefore \( I(x) = F(x) - F(a) \) and \( \int_{a}^{b} f(x) \, dx = I(b) = F(b) - F(a) \) as desired.\(^5\)

**EXAMPLE:** Evaluate the integral: \( \int_{-\sqrt{5}}^{\sqrt{5}} 5 + 4x^2 - x^4 \, dx \). Reversing our Derivative Rules as we did in §3.9, we see that \( F(x) = 5x + \frac{4}{3}x^3 - \frac{1}{5}x^5 \) is an antiderivative. By the Theorem:

\[
\int_{-\sqrt{5}}^{\sqrt{5}} 5 + 4x^2 - x^4 \, dx = F(\sqrt{5}) - F(-\sqrt{5}) = \frac{20}{3}\sqrt{5} - (-\frac{20}{3}\sqrt{5}) = \frac{40}{3}\sqrt{5} \approx 29.81
\]

**EXAMPLE:** Determine the area under the curve \( y = 5 + 4x^2 - x^4 \) and above the \( x \)-axis.

We must determine the limits of integration, which are the \( x \)-intercepts of the graph. Substituting \( u = x^2 \), the equation becomes \( 5 + 4u - u^2 = 0 \), which we can solve by the Quadratic Formula as \( u = -1 \) or \( 5 \), so \( x = \pm\sqrt{u} = \pm\sqrt{5} \). Thus the area is \( \int_{-\sqrt{5}}^{\sqrt{5}} 5 + 4x^2 - x^4 \, dx = \frac{40}{3}\sqrt{5} \).

*(Check: Graph’s average height \( \approx 7 \), base \( \approx 4 \), so area \( \approx 28 \), agreeing with above.)*

\(^5\)The variable of integration, \( x \) or \( t \), is irrelevant, provided it doesn’t conflict with the limits of integration.