Fundamental Theorem of Calculus

First Fundamental Theorem of Calculus. In §3.9, we introduced *algebraic* antiderivatives: given a function $f(x)$, we reverse derivative rules to guess a formula for the unique $F(x)$ with $F'(x) = f(x)$ and $F(a) = 0$. In §4.1, we formulated a *physical* meaning for the antiderivative as an integral $F(b) = \int_a^b f(x)\,dx$, the cumulative effect of the influence $f(x)$ acting from $x = a$ to $x = b$; and a *geometric* meaning as the area under the graph $y = f(x)$ and above the horizontal interval $x \in [a, b]$, with area below the x-axis counted negative. *Numerically*, we saw that cumulative effects and areas can be approximated by Riemann sums.

In §4.2, we started to put this complicated concept into logical order as an axiomatic theory. First, we identified the logical core of the concept, and formulated the bedrock definition of an integral as a limit of Riemann sums:

$$\int_a^b f(x)\,dx \stackrel{\text{def}}{=} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and the x_i's are any sample points in each x-increment. From this fundamental numerical definition, the physical and geometric meanings follow fairly obviously: the Riemann sum terms are the small increments of a cumulative effect, or the small vertical rectangles constituting an area.

In this section, we place the keystone in our axiomatic theory, showing that the numerical definition also implies our original meaning: integrals produce antiderivative functions.

> *Theorem:* Let $f(x)$ be continuous over $x \in [a, b]$, and define the function $I(x)$ that integrates f over a variable interval $t \in [a, x]$:*
>
> $$I(x) = \int_a^x f(t)\,dt.$$
>
> Then $I'(x) = f(x)$, and $I(x)$ is the unique antiderivative of $f(x)$ with $I(a) = 0$.

Physically this means: the rate of change of a cumulative effect up to some time is equal to the strength of the effect at that time.

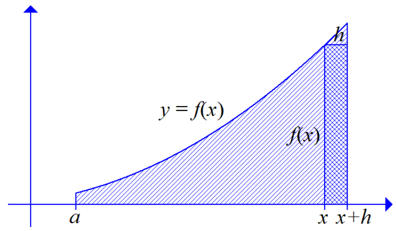
Proof. This is a rigorous argument intended to be so clear and certain that you can bet your life on its correctness in every case. For this, we cannot rely on our physical intuition about velocities and positions, because any picture only shows a particular case, and is not numerically precise. We do not even know for sure that there exists any anti-derivative function. Rather, we build on the bedrock definition to produce the candidate anti-derivative $I(x) = \int_a^x f(x)\,dx$ as a limit of Riemann sums. Then we try to differentiate using the numerical definition of derivative: $I'(x) = \lim\limits_{h\to 0} \frac{I(x+h)-I(x)}{h}$. We have:

$$\frac{I(x+h) - I(x)}{h} = \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right) = \frac{1}{h}\int_x^{x+h} f(t)\,dt\,,$$

since $\int_a^{x+h} = \int_a^x + \int_x^{x+h}$ for all h (even $h < 0$) by the Splitting Rule (§4.2).

Notes by Peter Magyar magyarp@msu.edu

*We must use different letters for the limit of integration x and the variable of integration t, since otherwise $I(x) \stackrel{??}{=} \int_a^x f(x)\,dx$ would imply nonsense like $I(2) \stackrel{???}{=} \int_a^2 f(2)\,d2$.



Cheating for a moment, we visualize this geometrically: if h is small enough, the region above $[x, x{+}h]$ is approximately a rectangle with (height)×(width) $= f(x)\,h$ and:

$$I'(x) \;\approx\; \frac{I(x{+}h) - I(x)}{h} \;=\; \frac{1}{h}\int_x^{x+h} f(t)\,dt \;\approx\; \tfrac{1}{h}(f(x)\,h) \;=\; f(x),$$

with approximations turning into equalities as $h \to 0$, so $I'(x) = f(x)$ as claimed by the Theorem. However, such geometric inspection is not enough for an axiomatic theory.

To control errors for sure, we take the absolute minimum value N and the absolute maximum value M of the continuous function $f(x)$ on $[x, x{+}h]$, using the Extremal Value Theorem (§3.1).† (To indicate that these depend on h, we write N_h, M_h.) Now, $N_h \le f(t) \le M_h$ for $t \in [x, x{+}h]$, so by the Bounds Rule for integrals (§4.2) we have:

$$((x{+}h){-}x)N_h \;\le\; \int_x^{x+h} f(t)\,dt \;\le\; ((x{+}h){-}x)M_h \quad\Longrightarrow\quad N_h \;\le\; \frac{1}{h}\int_x^{x+h} f(t)\,dt \;\le\; M_h.$$

As h gets very small, the interval $[x, x{+}h]$ gets closer and closer to the single point x, and the absolute minimum and maximum over this tiny interval must approach $f(x)$ by continuity: that is, $\lim_{h\to 0} N_h = \lim_{h\to 0} M_h = f(x)$. Also, by the above we have:

$$N_h \;\le\; \frac{I(x{+}h) - I(x)}{h} \;=\; \frac{1}{h}\int_x^{x+h} f(t)\,dt \;\le M_h\,.$$

Applying the Squeeze Theorem for limits (§1.6), we find what we wanted:

$$I'(x) \;=\; \lim_{h\to 0}\frac{I(x{+}h) - I(x)}{h} \;=\; \lim_{h\to 0} N_h \;=\; \lim_{h\to 0} M_h \;=\; f(x)\,,$$

As for the uniqueness part of the conclusion, it is clear that $I(a) = \int_a^a f(t)\,dt = 0$, and the Antiderivative Theorem (§3.9), a version of the Uniqueness Theorem (§3.2), guarantees there is a unique antiderivative with this initial value. Note how we have used almost all of our previous theory in proving this culminating Theorem.

Derivative of integral functions. The above Theorem can be stated as a Basic Derivative formula for $I(x) = \int_a^x f(t)\,dt$, where $f(t)$ is continuous:

$$I'(x) \;=\; \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) \;=\; f(x)\,.$$

Here a is any constant, x is the input variable, and t is a "dummy variable" which only has meaning inside the integral.

If we take the composition of $I(x)$ with any other function $g(x)$, the above Basic Derivative together with the Chain Rule (§2.5) implies:

$$\frac{d}{dx}\left(\int_a^{g(x)} f(t)\,dt\right) \;=\; I(g(x))' \;=\; I'(g(x))\cdot g'(x) \;=\; f(g(x))\cdot g'(x)\,.$$

†Here we assume $h > 0$. The case $h < 0$ is the same except for a few sign changes.

EXAMPLE: Find the derivative of $F(x) = \int_{2x}^{x^3} \sin(x)\, dx$. We have:

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left(\int_{2x}^{x^3} \sin(x)\, dx\right) = \frac{d}{dx}\left(\int_0^{x^3} \sin(x)\, dx - \int_0^{2x} \sin(x)\, dx\right) \\ &= \sin(x^3)\cdot(x^3)' - \sin(2x)\cdot(2x)' = 3x^2\sin(x^3) - 2\sin(2x)\,. \end{aligned}$$

EXAMPLE: Find the derivative of $F(x) = \int_{2a}^{b^3} \sin(t)\, dt$. Here a, b are constants, and hence so are $2a, b^3$. In fact, the right hand side does not depend on the variable x, and is a constant function with derivative $F'(x) = 0$! This also follows from the Chain Rule, since $\sin(2a)\cdot 2(a)' = 0$ and $\sin(b^3)\cdot(b^3)' = \sin(b^3)\cdot 3b^2\cdot(b)' = 0$.

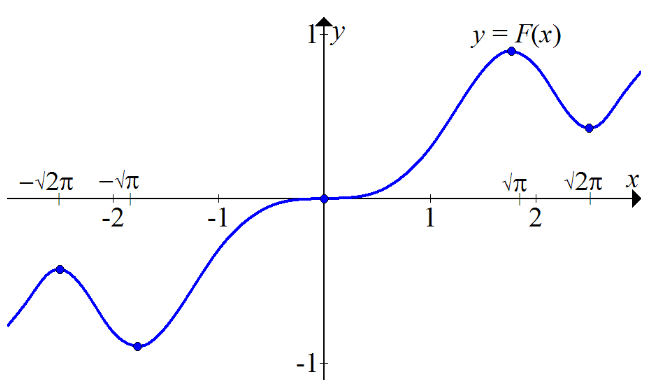
Sketching integral functions. Since an antiderivative $I(x) = \int_a^x f(t)\, dt$ might be a completely new function for which no elementary formula is possible, it might seem mysterious. However, we can compute its values with sufficient numerical accuracy using Riemann sums, and plot these to draw the graph.

A geometric strategy is to use the derivative $I'(x) = f(x)$ for sketching $y = I(x)$, as in §3.3 and §3.5. That is, the *slope* of the graph $y = I(x)$ is given by the *height* of $y = f(x)$.

EXAMPLE: Graph the function $I(x) = \int_0^x \sin(t^2)\, dt$. The critical points of $I(x)$ are those x where $I'(x) = 0$, i.e. $f(x) = \sin(x^2) = 0$. (There are no critical points where $I'(x)$ is undefined.) This happens when $x^2 = 2k\pi$ for any integer k, so the critical points are $x = 0, \pm\sqrt{\pi}, \pm\sqrt{2\pi}, \ldots$. Sign chart:

x		$-\sqrt{2\pi}$		$-\sqrt{\pi}$		0		$\sqrt{\pi}$		$\sqrt{2\pi}$	
$I'(x)$	$+$	0	$-$	0	$+$	0	$+$	0	$-$	0	$+$
$I(x)$	↗	-0.43	↘	-0.89	↗	0	↗	0.89	↘	0.43	↗

For inflection points, we solve $I''(x) = 0$, i.e. $f'(x) = 2x\cos(x^2) = 0$, so $x = 0, \pm\sqrt{\frac{\pi}{2}}$, $\pm\sqrt{\frac{3\pi}{2}}, \ldots$. This is enough data for a detailed graph:



From the 180° rotational symmetry of the graph, we see $I(x)$ is an odd function, $I(-x) = -I(x)$, which happens for the integral of any even function, $f(-x) = f(x)$; see §4.5.

Second Fundamental Theorem. This is a trick to easily evaluate many integrals, which we already used to find some exact values in §4.1.

> *Theorem:* Suppose $F(x)$ is some known antiderivative with $F'(x) = f(x)$. Then:
>
> $$\int_a^b f(x)\,dx \;=\; F(b) - F(a)\,.$$
>
> That is, if $f(x)$ is the rate of change of $F(x)$, then the integral $\int_a^b f(x)\,dx$ is equal to the total change of $F(x)$ from $x = a$ to b.

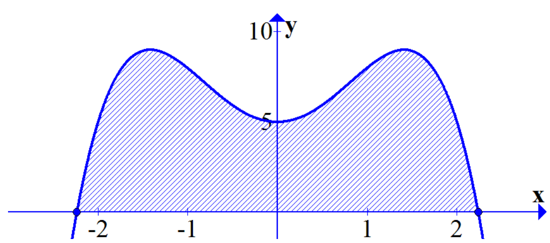
Physically this means: the cumulative effect of a rate of change is the total change.

Proof. Since $F(x)$ is a particular antiderivative of $f(x)$, the Uniqueness Theorem (§3.9, §3.2) says that the general antiderivative is $F(x)+C$ for any constant C. But the First Fundamental Theorem says the integral function $I(x) = \int_0^x f(t)\,dt$ is also an antiderivative of $f(x)$, so we must have $I(x) = F(x) + C$. Since we know the initial condition $I(a) = \int_a^a f(t)\,dt = 0$, we get $I(a) = F(a) + C = 0$, and $C = -F(a)$. Therefore $I(x) = F(x) - F(a)$ and $\int_a^b f(t)\,dt = I(b) = F(b) - F(a)$ as desired.[‡]

EXAMPLE: Evaluate the integral: $\int_{-\sqrt{5}}^{\sqrt{5}} 5{+}4x^2{-}x^4\,dx$. Reversing our Derivative Rules as we did in §3.9, we see that $F(x) = 5x + \frac{4}{3}x^3 - \frac{1}{5}x^5$ is an antiderivative. By the Theorem:

$$\int_{-\sqrt{5}}^{\sqrt{5}} 5{+}4x^2{-}x^4\,dx \;=\; F(\sqrt{5}) - F(-\sqrt{5}) \;=\; \tfrac{20}{3}\sqrt{5} - (-\tfrac{20}{3}\sqrt{5}) \;=\; \tfrac{40}{3}\sqrt{5} \;\approx\; 29.81$$

EXAMPLE: Determine the area under the curve $y = 5 + 4x^2 - x^4$ and above the x-axis.



We must find the limits of integration, which are the x-intercepts of the graph. Substituting $u = x^2$, the equation becomes $5{+}4u{-}u^2 = 0$, which we can solve by the Quadratic Formula as $u = -1$ or 5, so $x = \pm\sqrt{u} = \pm\sqrt{5}$. Thus the area is $\int_{-\sqrt{5}}^{\sqrt{5}} 5{+}4x^2{-}x^4\,dx = \frac{40}{3}\sqrt{5}$. (Check: Graph's average height ≈ 7, base ≈ 4, so area ≈ 28, consistent with the above.)

EXAMPLE: Find $\int_{-1}^{3} |x|\,dx$. This can be done geometrically by finding the area of the two triangles under the graph. Algebraically, the integrand $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$ has anti-derivative $F(x) = \begin{cases} x^2/2 & \text{if } x \geq 0 \\ -x^2/2 & \text{if } x \leq 0\,. \end{cases}$ Thus:

$$\int_{-1}^{3} |x|\,dx \;=\; F(3) - F(-1) = \tfrac{3^2}{2} - (-\tfrac{(-1)^2}{2}) \;=\; \tfrac{10}{2} \;=\; 5\,.$$

[‡] Again, the variable of integration, t or x, is irrelevant unless it conflicts with the limits of integration.