Sigma Notation

Notation for sums. In Notes §4.1, we define the integral $\int_a^b f(x)\,dx$ as a limit of approximations. That is, we split the interval $x \in [a,b]$ into n increments of size $\Delta x = \frac{b-a}{n}$, we choose sample points $x_1, x_2, \ldots, x_n$, and we take:

$$\int_a^b f(x)\,dx \quad = \quad \lim_{n\to\infty} f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

The sum which appears on the right is called a *Riemann sum.* Similar sums appear frequently in mathematics, and we define a special notation to handle them.

In the most general situation, we have a sequence of numbers $q_0, q_1, q_2, q_3, \ldots$ so that for any $i = 0, 1, 2, 3, \ldots$ we have a number q_i. We consider an interval of integers $i = m, m{+}1, m{+}2, \ldots, n$, and we introduce a notation for the sum of all the q_i for $i = m$ to n:

$$\sum_{i=m}^{n} q_i \;=\; q_m + q_{m+1} + q_{m+2} + \cdots + q_n.$$

The summation symbol Σ is capital sigma, the Greek letter S meaning *sum.* The variable i is called the *index of summation.*

Examples

- Letting $q_i = \sqrt{i}$, we have $q_0 = \sqrt{0} = 0$, $q_1 = \sqrt{1} = 1$, $q_2 = \sqrt{2}$, $q_3 = \sqrt{3}$, $\ldots$, and taking the interval of integers $i = 2, 3, 4, 5$, we have:

$$\sum_{i=2}^{5} \sqrt{i} \;=\; \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} \;\approx\; 7.38\,.$$

- Letting $q_i = 1$, we have: $\sum_{i=1}^{10} 1 = \underbrace{1 + 1 + \cdots + 1}_{10 \text{ terms}} = 10$.

- Given the sum of the first ten square numbers $1 + 4 + 9 + 16 + \cdots + 100$, we wish to write this compactly in sigma notation. Considering the terms as a sequence $q_i = i^2$, we get:

$$1 + 4 + 9 + \cdots + 100 \;=\; 1^2 + 2^2 + 3^2 + \cdots + 10^2 \;=\; \sum_{i=1}^{10} i^2\,.$$

- Given the sum of the first five odd numbers $1 + 3 + 5 + 7 + 9$, we can write this in sigma notation by considering the terms as $q_i = 2i{-}1$:

$$1 + 3 + 5 + 7 + 9 \;=\; (2(1){-}1) + (2(2){-}1) + \cdots + (2(5){-}1) \;=\; \sum_{i=1}^{5} (2i{-}1)\,.$$

  Another way would be to consider the terms as $q_i = 2i{+}1$:

$$1 + 3 + 5 + 7 + 9 \;=\; (2(0){+}1) + (2(1){+}1) + \cdots + (2(4){+}1) \;=\; \sum_{i=0}^{4} (2i{+}1)\,.$$

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- The sum of the first n odd numbers, where n is an unspecified whole number:

$$1+3+5+\cdots+(2n{-}1) \;=\; \sum_{i=1}^{n}(2i{-}1).$$

- For a sum with alternating plus and minus terms, use $(-1)^{i-1}=\begin{cases}1 & \text{for odd } i\\ -1 & \text{for even } i.\end{cases}$

$$1-3+5-7+\cdots\pm(2n{-}1) \;=\; \sum_{i=1}^{n}(-1)^{i-1}(2i{-}1).$$

- Riemann sum for $\int_a^b f(x)\,dx$ with $\Delta x=\frac{b-a}{n}$ and sample points $x_i=a{+}i\Delta x$:

$$f(x_1)\Delta x{+}f(x_2)\Delta x{+}\cdots{+}f(x_n)\Delta x \;=\; \sum_{i=1}^{n} f(x_i)\Delta x \;=\; \sum_{i=1}^{n} f\left(a+\tfrac{i}{n}(b-a)\right)\tfrac{b-a}{n}\,.$$

Summation Rules. As we did for limits and derivatives, we can sometimes compute summations by starting with known Basic Summations, and combining them by Summation Rules.

- *Sum:* $\displaystyle\sum_{i=m}^{n}(q_i{+}p_i) \;=\; \sum_{i=m}^{n} q_i \;+\; \sum_{i=m}^{n} p_i\,.$
- *Difference:* $\displaystyle\sum_{i=m}^{n}(q_i{-}p_i) \;=\; \sum_{i=m}^{n} q_i \;-\; \sum_{i=m}^{n} p_i\,.$
- *Constant Multiple:* $\displaystyle\sum_{i=m}^{n} C\,q_i \;=\; C\cdot\sum_{i=m}^{n} q_i$, where C does not depend on i.

Like all facts about summations, these formulas can be understood by writing out the terms in dot-dot-dot (ellipsis) notation, for example:

$$\begin{aligned}\sum_{i=m}^{n}(q_i{+}p_i) &= (q_m{+}p_m)+(q_{m+1}{+}p_{m+1})+\cdots+(q_n{+}p_n)\\ &= (q_m+q_{m+1}+\cdots+q_n)+(p_m+p_{m+1}+\cdots+p_n)\\ &= \sum_{i=m}^{n} q_i \;+\; \sum_{i=m}^{n} p_i\,.\end{aligned}$$

Note that n is a constant not depending on i, so we may factor it out of a summation: $\sum_{i=1}^{n} ni^2 = n\sum_{i=1}^{n} i^2$. Specifically for $n=3$, this means $3(1^2)+3(2^2)+3(3^2)=3(1^2{+}2^2{+}3^2)$. However, the variable i has *no meaning* outside the summation, and cannot be factored out: $\sum_{i=1}^{3} i2^i \stackrel{??}{=} i\sum_{i=1}^{3} 2^i$ is nonsense: the left side means $1(2^1)+2(2^2)+3(2^3)$, but the right side should mean a constant i times $2^1{+}2^2{+}2^3$, but i is *not* a constant.

Warning: the summation of a product $\sum q_i p_i$ is NOT equal to the product of summations $(\sum q_i)(\sum p_i)$. For example: $1\cdot1+2\cdot2+3\cdot3 \neq (1{+}2{+}3)(1{+}2{+}3)$.

Basic Summations. We can get a few surprisingly neat formulas:

(a) $$\sum_{i=1}^{n} 1 \;=\; n;$$

(b) $$\sum_{i=1}^{n} i \;=\; 1+2+\cdots+n \;=\; \tfrac{1}{2}n(n{+}1);$$

(c) $$\sum_{i=1}^{n} i^2 \;=\; 1^2+2^2+\cdots+n^2 \;=\; \tfrac{1}{6}n(n{+}1)(2n{+}1).$$

Proof. (a) $\sum_{i=1}^{n} 1 = 1+\cdots+1$ with n terms, which indeed equals n.

(b) Taking two copies of $\sum_{i=1}^{n} i$, we can pair each term with its complement:

$$\begin{array}{rcccccccccccc} 2\cdot\sum_{i=1}^{n} i & = & & 1 & + & 2 & + & \cdots & + & n{-}1 & + & n & \\ & & + & n & + & n{-}1 & + & \cdots & + & 2 & + & 1 & \\ \hline & & = & n{+}1 & + & n{+}1 & + & \cdots & + & n{+}1 & + & n{+}1 & = \; n(n{+}1). \end{array}$$

The equation $2\cdot\sum_{i=1}^{n} i = n(n{+}1)$, divided by 2, gives the desired formula.

(c) Consider that $(i{+}1)^3 = i^3 + 3i^2 + 3i + 1$, so that:

$$\begin{aligned} \textstyle\sum_{i=1}^{n} (i{+}1)^3 - i^3 \;&=\; \textstyle\sum_{i=1}^{n} (3i^2+3i+1) \\ &=\; \textstyle 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \\ &=\; \textstyle 3\sum_{i=1}^{n} i^2 + \frac{3}{2}n(n{+}1) + n. \end{aligned}$$

On the other hand, we have a "collapsing sum":

$$\begin{aligned} \textstyle\sum_{i=1}^{n} (i{+}1)^3 - i^3 \;&=\; (n{+}1)^3 {-} n^3 \;+\; n^3 {-} (n{-}1)^3 \;+\; \cdots \;+\; 3^3{-}2^3 \;+\; 2^3{-}1^3 \\ &=\; (n{+}1)^3 - 1^3\,. \end{aligned}$$

Solving the equation:

$$3\cdot\sum_{i=1}^{n} i^2 + \tfrac{3}{2}n(n{+}1) + n \;=\; (n{+}1)^3 - 1$$

gives, as desired:

$$\sum_{i=1}^{n} i^2 \;=\; \tfrac{1}{3}\big((n{+}1)^3 - \tfrac{3}{2}n(n{+}1) - (n{+}1)\big) \;=\; \tfrac{1}{6}n(n{+}1)(2n{+}1)\,.$$

A similar computation will produce a formula for $\sum_{i=1}^{n} i^3$, etc.