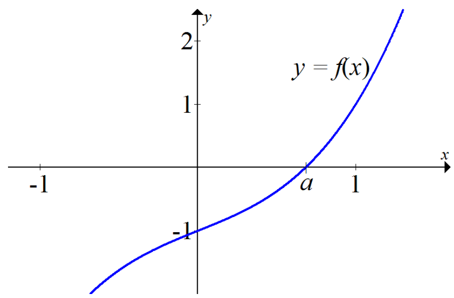
Roots of equations. We frequently need to solve equations for which there is no neat algebraic solution, such as:

$$f(x) \;=\; x^3 + x - 1 \;=\; 0\,.$$

In this case, the best we can ask is an approximate solution, accurate to a specified number of decimal places, and this is all we need for any practical purpose.

We can start with a computer graph of $y = f(x)$, which is just a display of many plotted points $(x, f(x))$:



A solution of $f(x) = 0$ is an x-intercept of the graph, and we see one,* call it $x = a$, close to $x = 0.5$; that is, our first estimate is $a \approx 0.5$. Computing:

$$f(0.5) = -0.375 < 0, \quad f(0.6) = -0.184 < 0, \quad f(0.7) = 0.043 > 0,$$

the Intermediate Value Theorem (§1.8) guarantees a solution $0.6 < a < 0.7$; thus we can improve our estimate to $a \approx 0.6$. We could add a decimal place by checking $f(0.61), f(0.62), \ldots, f(0.69)$ to see where the values change from negative to positive, but this is clearly very tedious and inefficient.

Newton's Method is an amazingly efficient way to refine an approximate solution to get more and more accurate ones, until the required accuracy is reached. Let us call our first estimate $x_1 = 0.5$. We are seeking the true solution $x = a$, the x-intercept of $y = f(x)$. As in §2.9, let us approximate $y = f(x)$ by its tangent line at our initial point at $(x_1, f(x_1))$, namely $y = f(x_1) + f'(x_1)(x{-}x_1)$:

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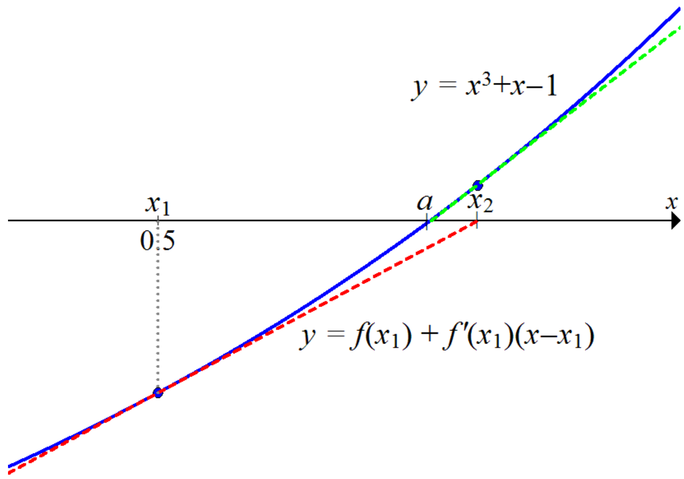
* How do we know there is no other solution $x = b$? If there were, Rolle's Theorem (§3.2) says that there would be some $x = c \in (a, b)$ with $f'(c) = 0$, namely a hill or valley of $y = f(x)$. But $f'(x) = 3x^2 + 1 = 0$ clearly has no solutions, so $y = f(x)$ has no hills or valleys, and there cannot exist another solution $x = b$.



You can see how the tangent line (in red) is very close to the graph near $x = x_1$, and fairly close even near the true solution $x = a$. We cannot solve for the x-intercept of $y = f(x)$, but we can find the x-intercept of the line, denoted $x = x_2$:

$$f(x_1) + f'(x_1)(x - x_1) \;=\; 0 \qquad \Longrightarrow \qquad x \;=\; x_2 \;=\; x_1 - \frac{f(x_1)}{f'(x_1)}\,.$$

This solution x_2 is not exactly a, but it is closer than the initial estimate x_1.

Now we can iterate (green line), repeating the same computation starting with x_2 instead of x_1. The result is:

$$x_3 \;=\; x_2 - \frac{f(x_2)}{f'(x_2)}\,,$$

which is much closer to a; then $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$; and repeating the same way we get the following spreadsheet, computing to 3 decimal places:

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
1	0.500	−0.375	1.750	0.714
2	0.714	0.079	2.531	0.683
3	0.683	0.002	2.400	0.682
4	0.682	0.000	2.397	0.682
5	0.682	0.000	2.397	0.682

The x_n's will continue as real numbers to converge closer and closer to a, but since we do not see any difference in our 3 decimal places after x_4, there is no point in continuing. We already have our answer within the specified accuracy:

$$a \approx 0.682 \quad \text{accurate to 3 decimal places.}$$

In the table, $f(x_4) \approx 0.000$ is indeed an approximate solution to $f(x) = 0$.

Newton's Method. We wish to solve an equation $f(x) = 0$, with the true solution $x = a$ fairly close to an intial estimate $a \approx x_1$, and the final approximation $a \approx x_n$ accurate to a specified number of decimal places.

1. Using a calcuator, spreadsheet, or computer algebra system, compute $x_2, x_3, \ldots$ according to the formula:

$$x_{n+1} \;=\; x_n - \frac{f(x_n)}{f'(x_n)},$$

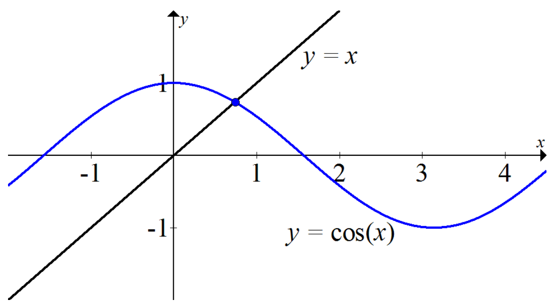
computing with *at least* the specified accuracy (number of decimal places).

2. Stop once $x_n \approx x_{n+1}$ are the same up to the given accuracy. The final approximation is $a \approx x_n$.

Trigonometric equation: Solve the following equation to 3 decimal places:

$$\cos(x) = x\,.$$

(As always in Calculus, we assume x is in radians: see §2.5 end.)



Looking at the graph, we see that there is a unique solution somewhere around $x_1 = 1$. This seems different from the previous case, since we seek the intersection of two graphs rather than the x-intercept of a single graph; but we can simply rewrite the equation as $f(x) = x - \cos(x) = 0$. Newton's Method gives:

$$x_{n+1} \;=\; x_n - \frac{x_n - \cos(x_n)}{1+\sin(x_n)}\,,$$

x_1	x_2	x_3	x_4
1.000	0.750	0.739	0.739

That is, the solution is $a \approx 0.739$ to 3 places.

Numerical roots. The number $\sqrt{2}$ is a "known value": a calculator can immediately tell us that $\sqrt{2} = 1.41421356\ldots$. But just how does the calculator know this? Newton's Method, that's how!

By definition, $\sqrt{2}$ is the solution of $x^2 = 2$, or $f(x) = x^2 - 2 = 0$. Starting with $x_1 = 1$, the Method gives $x_{n+1} = x_n - \frac{x_n^2-2}{2x_n}$ and:

x_1	1.00000000
x_2	1.50000000
x_3	1.41666667
x_4	1.41421569
x_5	1.41421356
x_6	1.41421356

Here we see the power of the Method: with just a couple of dozen $+, -, \times, \div$ calculator operations, it converged from 0 places to 8 places of accuracy.

We could also do the Method with fractions rather than decimals to get very accurate fractional approximations of $\sqrt{2}$:

x_1	1
x_2	3/2
x_3	17/12
x_3	577/408

Already $x_3 = \frac{17}{12}$ is a very good approximation, since $(\frac{17}{12})^2 = \frac{289}{144} = 2\frac{1}{144}$, very close to 2. However, no fraction or finite decimal can give $\sqrt{2}$ exactly: it is known to be an *irrational* number.

Rate of convergence. Newton's Method gives great accuracy very quickly. In fact, each iteration approximately *doubles* the number of accurate decimal places. We discuss approximation errors in Calculus II §11.11.