Math 132 Overview

This section is a bird's-eye view of the course. Read it over now, then come back to it as you learn the topics, to see how they fit into the whole theory.

Calculus is the mathematics of change and variation. With ordinary algebra, we can translate static or linear real-world problems into equations and solve them; with calculus, we can solve dynamic problems involving non-linear motion, varying rates of change, optimum values, curved shapes, and the cumulative effect of a changing influence. It was discovered by Newton and Leibnitz, and developed further notably by Euler and Riemann.

The main concepts of calculus are *derivatives* and *integrals* applied to *functions*. Like most mathematical concepts, these have four levels of meaning: physical (real-world), geometric (pictures), numerical (spreadsheets), and algebraic (formulas). Given a problem originating on one level (usually physical or geometric), we translate to a different level (numerical or algebraic) where the problem can be solved, then we translate the solution back to the original level.

Functions. Officially, a function $f : X \to Y$ is any rule that takes elements of an input set X (the domain) to elements of an output set Y. In problems, this concept is represented on the following levels.

1. Physical: A function defines how an input quantity (the independent variable or argument) determines an output quantity (the dependent variable or value). For example, consider a stone dropped from a bridge: the elapsed time t (in sec) determines the observed distance s (in feet) that the stone has fallen, $s = f(t)$. The initial value is $f(0) = 0$, and if the stone falls into the water 400 ft below after 5 sec, then $f(5) = 400$ and the domain is naturally $0 \leq t \leq 5$, namely the interval $X = [0, 5]$.

2. Geometric: A function is a graph in the plane, the curve of points (x, y) such that $y = f(x)$. In our example, we use coordinates (t, s), and the graph $s = f(t)$ curves upward from $(0, 0)$ to $(5, 400)$. As the stone speeds up with increasing t, the graph gets steeper: in fact, it is a segment of a parabola.

3. Numerical: A function is a table of values. In our example, we might get a partial such table by measuring the distance at sample times:

t	0	1	2	3	4	5
$s{=}f(t)$	0	16	64	144	256	400

   Of course, $f(t)$ has a value at every t, not just the samples. We can imagine the full function as an infinite table with an entry for every t in the domain.

4. Algebraic: A function is a formula to compute the output in terms of the input. A model of our physical example is the formula $f(t) = 16t^2$. Like all models of the real world, this is accurate only within a bounded domain ($0 \leq t \leq 5$) and up to some error (from air resistance or imprecise measurements).

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Derivatives. Now we preview the main concepts of this course. Given a function f, the *derivative function* f' has the following meanings.

1. Physical: The derivative of a function $y = f(x)$ is the rate of change of y with respect to the change in x. In our example of a falling stone, $s = f(t)$, the derivative tells how fast the distance is increasing per unit time, i.e. how fast the stone is moving in feet per second. This is the instantaneous *velocity* v at time t, so the derivative is the velocity function $v = f'(t)$.

2. Geometric: For a graph $y = f(x)$, the derivative $f'(a)$ at $x = a$ is the slope of the graph near the point $(a, f(a))$: that is, the slope of the tangent line at that point, $y = f(a) + m(x-a)$, where $m = f'(a)$.

   Maximum and minimum heights (hills/valleys) of the graph occur at *critical points* $(a, f(a))$ having horizontal tangent $f'(a) = 0$ (or $f'(a)$ undefined).

3. Numerical: We can approximate the derivative $f'(a)$, instantaneous velocity, by considering an input $x = a + h$ close to a, and dividing the rise in $f(x)$ by the run in x:

   $$f'(a) \;\approx\; \frac{\Delta f}{\Delta x} \;=\; \frac{f(x) - f(a)}{x - a} \;=\; \frac{f(a{+}h) - f(a)}{h}\,.$$

   In our example $f(t) = 16t^2$, we can compute the approximate velocity at the instant $t = 3$ sec by considering the short time interval $3 \le t \le 3.1$, and computing the distance traveled (change in distance), divided by the time elapsed:

   $$v = f'(3) \;\cong\; \frac{f(3.1) - f(3)}{3.1 - 3} \;=\; \frac{153.76 - 144}{0.1} \;=\; 97.6\;.$$

   That is, after falling for 3 sec, the stone is travelling at about 97.6 ft/sec.

   Once we know $f'(a)$, we can use it to approximate $f(x)$ by a linear function $f(x) \approx f(a) + f'(a)(x-a)$ for x near a, with error sensitivity $\Delta f \approx f'(a)\,\Delta x$.

4. Algebraic: We will give methods to compute the derivative of any formula. The foundation is the precise definition: the derivative of $f(x)$ at $x = a$ is the limiting value of its rate of change over a short interval $a \le x \le a{+}h$ as the width $\Delta x = h$ becomes smaller and smaller toward zero ($h \to 0$):

   $$f'(a) \;=\; \lim_{h\to 0} \frac{f(a{+}h) - f(a)}{h}\,.$$

   We use this limit definition to determine some Basic Derivatives such as $(x^p)' = px^{p-1}$, $\sin'(x) = \cos(x)$, $\cos'(x) = -\sin(x)$, and then Rules for combining them: Sum $(f{+}g)' = f'{+}g'$, Product $(fg)' = f'g{+}fg'$, Quotient $(f/g)' = (f'g - fg')/g^2$, and Chain or composition $f(g(x))' = f'(g(x))\,g'(x)$.

   For our example $f(t) = 16t^2$, we get $f'(t) = 32t$: the velocity is steadily increasing proportional to time. The exact modeled velocity is $f'(3) = 96$.

Integrals. Given a function g, its *integral* from $x = a$ to $x = b$ is a number denoted $\int_a^b g(x)\,dx$, and has the following meanings.

1. Physical. Suppose a quantity $z = f(x)$ is influenced linearly by another function $g(x)$ as the input goes from $x = a$ to $x = b$: i.e. each incremental change Δx leads to a small change $\Delta z \approx g(x)\,\Delta x$. Then the integral of $g(x)$ is the cumulative effect of $g(x)$, the total change in z from $x = a$ to $x = b$:

$$\int_a^b g(x)\,dx = f(b) - f(a).$$

   In our example, suppose we start by knowing the velocity of the stone, $v = g(t) = 32t$, and we wish to deduce the distance fallen, $s = f(t)$ for $t = 3$. Over a time increment Δt, the stone moves by about $\Delta s \approx v\,\Delta t = 32t\,\Delta t$; so we can express the cumulative change as: $f(3) = f(3) - f(0) = \int_0^3 32t\,dt$.

2. Geometric. For the graph $y = g(x) \geq 0$, the integral $\int_a^b g(x)\,dx$ is the area under the graph and above the interval $a \leq x \leq b$ on the x-axis. This is because the area A is the cumulative total of thin slices $\Delta A \approx g(x)\,\Delta x$ with height $y = g(x)$ and width Δx. (Area under the x-axis is counted negative.)

   In our example, we can get the integral $\int_0^3 32t\,dt$ as the triangular area under the graph $v = g(t) = 32t$ and above $t \in [0, 3]$: i.e. $\int_0^3 32t\,dt = \frac{1}{2}(3)(96) = 144$.

3. Numerical. We approximate the cumulative effect of $g(x)$ from $x = a$ to $x = b$ by splitting up the interval $a \leq x \leq b$ into a large number n of small increments of width $\Delta x = \frac{b-a}{n}$. We take sample points $x_1, \ldots, x_n$, one in each increment, and compute the "Riemann sum" of all $\Delta z \approx g(x_i)\Delta x$:

$$\int_a^b g(x)\,dx \;\approx\; g(x_1)\Delta x + g(x_2)\Delta x + \cdots + g(x_n)\Delta x\,.$$

   This is the origin of the notation $\int_a^b g(x)\,dx$, where $\int$ is an elongated S standing for "sum," and $g(x)\,dx$ represents all the small changes $g(x_i)\Delta x$.

   In our example, given the velocity function $v = g(t) = 32t$, we can take $n = 3$, $\Delta t = 1$ sec, and sample points $t_1{=}1$, $t_2{=}2$, $t_3{=}3$. We approximate cumulative distance traveled by computing (velocity at t_i)$\times$(time elapsed) $= 32t_i\,\Delta t$ for each $i = 1, 2, 3$, and adding these terms:

$$\int_0^3 32t\,dt \;\approx\; 32(1)(1) + 32(2)(1) + 32(3)(1) \;=\; 192.$$

   This overestimates because we sample the velocity at the end of each time increment, when the stone is fastest. Taking more increments (larger n) gives better and better approximations whose limit is the exact integral.

4. Algebraic. Since integrals go from a rate of change to a total change, they are reverse derivatives (antiderivatives), and we can use our known derivative rules backwards to find formulas for many (but not all) integrals. That is, if $g(x) = f'(x)$ for a known formula $f(x)$, then $\int_a^b g(x)\,dx = f(b) - f(a)$. This is known as the *Second Fundamental Theorem of Calculus*.

   In our example, given $v = 32t$, we can find $f(t) = 16t^2$ with $f'(t) = 32t$, so we get the exact integral value $\int_0^3 32t\,dt = f(3) - f(0) = 16(3^2) - 16(0^2) = 144$.

What does it mean?
Newton discovered it in a country garden hiding out from the plague in 1666,
consummating five thousand years of math going back
to the first Sumerian nerds who scratched farmland measurements
and cattle tallies and quadratic problems onto clay tablets,
to the Egyptian priests who computed the slopes of tombs for their god king,
to Pythagoras who had a vision of numbers and shapes as the one ultimate reality,
to Euclid who built a soaring tower of theorems unshakably founded on axioms,
to Brahmagupta who grasped Nothing as a number, not an absence of number,
to Al-Khwarizmi the Persian who explained the solution of equations
by Qabalah and Algebra, Breaking and Mending,
to Descartes who in the modern sunrise looked at numbers and shapes as if for
the first time and could at last see how they describe the same deep thing.
Then Newton added that smooth functions and shapes are infinitesimally linear,
and accumulating linear increments is the inverse of taking linear rates,
insights deeper and more powerful than any before, Promethean fire
that burst open the gates to theoretical science and the Modern World.
Now if we can only learn some humility before it burns us up.