Principal Moments of Inertia

The *moment of inertia* $I_u$ of a solid body $V$ rotating about an axis through the origin in the direction given by the unit vector $u$ (counter clockwise when viewed point-on) is the scalar
\[
I_u = \int_V r^2 \, dm = \int_V (x \times u)^2 \, dm = \int_V (x \times u)^2 \rho \, dx,
\]
where $\rho = \rho(x)$ is mass density. The rotational kinetic energy of this body is then
\[
T = \frac{\omega^2 I_u}{2},
\]
where $\omega$ is the angular velocity of rotation.

By carrying out the integrations in (1) this moment of inertia is realized by a quadratic form:
\[
I_u = u^T A u,
\]
where $A^T = A$. By the spectral theorem we may perform a distance-preserving change of coordinates $P = (u_1, u_2, u_3)$, $P^T P = I$, so that the form (3) giving the moment of inertia (1) is now diagonal:
\[
I_{u'} = u'^T \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} u'
\]
with $I_1 \geq I_2 \geq I_3 > 0$. The eigenvalues $I_1, I_2, I_3$ are called the *principal moments of inertia*. Their associated eigenvectors $u_1, u_2, u_3$ are called the *principal axes.*

**Example.** Think about an American football: One principal axis $u_3$ is along the long axis, and the other two principal axes $u_1, u_2$ are in the 'equatorial plane,' with $I_1 = I_2 > I_3$.

**The Charon Principal.** If $I_1 > I_2 > I_3$, then rotation about the intermediate axis is unstable—asteroids tumble! (See Ian Stewart.)

**Question.** Are we sure that the orthogonal change of coordinates $P$ has preserved the right-handedness of the coordinate system?

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1This is historically the form in which the spectral theorem first appeared.
Fact. Suppose we have brought the symmetric $3 \times 3$ matrix $A$ to diagonal form:

$$P^{-1}AP = \text{diag}(\lambda_1, \lambda_2, \lambda_3) = \Lambda,$$

where $P$ is orthogonal, i.e., $P^{-1} = P^T$. Then there is a more careful choice of $P$ that is a rotation about some axis.

Proof. We may exchange two columns of $P$, if necessary, to ensure that $P$ has determinant 1. Swapping two columns of $P$ only changes the order of two of the eigenvalues $\lambda_i$ along the diagonal of $\Lambda$. In symbols, if $E$ is the elementary matrix swapping the two columns, then because $AP = P\Lambda$, we have $APE = P\Lambda E = PE(E^T \Lambda E)$.

Because $P$ is orthogonal, all its eigenvalues lie on the unit circle. Since $n = 3$, one or all three must be real. But since $P$ has determinant 1, one eigenvalue belonging to a unit eigenvector $w_0$ must be 1. We now show $P$ is rotation about $w_0$.

Because $P$ is orthogonal, it is normal, and hence eigenspaces of $P$ are reducing. In particular, the subspace $S$ orthogonal to $w_0$ is invariant under $P$. Find two perpendicular unit vectors $u_0, v_0 \in S$. Then with respect this basis $P$ has matrix

$$O^{-1}PO = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix},$$

where the lower right $2 \times 2$ block $Q$ has determinant 1.

The orthogonal map $P$ when cut back to $S$ is again orthogonal, hence

$$Q^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = Q^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$ 

Hence $a = d$ and $b = -c$ giving $1 = ad - bc = a^2 + b^2$. Thus $a = \cos \theta$ and $b = -\sin \theta$. Thus $P$ cut back to the plane $S$ is a rotation. Hence $P$ is a rotation about $w_0$. 
