Examples of Graphing Rational Functions

Here are two examples to illustrate graphing rational functions further. We saw a few examples in class and these are supposed to augment those.

**Example:** Graph the rational function

\[ y = f(x) = \frac{2x^2 + x - 1}{x^2 - 1}. \]

First note that the given function is neither odd nor even. So there is no associated symmetry in the graph.

**Asymptotes:**

*Vertical Asymptotes:* The only possibilities are when \( x^2 = 1 \). This happens when \( x = 1 \) and \( x = -1 \).

\( x = 1; \)

\[ \lim_{x \to 1^+} \frac{2x^2 + x - 1}{x^2 - 1} = \infty, \quad \lim_{x \to 1^-} \frac{2x^2 + x - 1}{x^2 - 1} = -\infty \]

So \( x = 1 \) is a vertical asymptote. (Note that for this to happen we needed only one of the limits to be \( \pm \infty \). But it is useful and necessary to know these limits when sketching the graph.)

\( x = -1; \)

Note that when \( x = -1 \), \( 2x^2 + x - 1 = 2(-1)^2 + (-1) - 1 = 0 \). So the numerator in \( f(x) \) has a factor of \( (x + 1) \). In fact,

\[ 2x^2 + x - 1 = (x + 1)(2x - 1). \]

So

\[ \lim_{x \to -1^-} \frac{2x^2 + x - 1}{x^2 - 1} = \lim_{x \to -1^+} \frac{2x^2 + x - 1}{x^2 - 1} = \lim_{x \to -1} \frac{2x^2 + x - 1}{x^2 - 1} \]

and these are equal to

\[ \lim_{x \to -1} \frac{(x + 1)(2x - 1)}{(x - 1)(x + 1)} = \lim_{x \to -1} \frac{2x - 1}{x - 1} = \frac{3}{2} \]
So $x = -1$ is not a vertical asymptote and $y = f(x)$ has a removable discontinuity at $x = -1$.

**Horizontal Asymptotes:** We compute the limits as $x \to \pm \infty$.

$$
\lim_{x \to \infty} \frac{2x^2 + x - 1}{x^2 - 1} = \lim_{x \to \infty} \frac{2 + \frac{1}{x} - \frac{1}{x^2}}{1 - \frac{1}{x^2}} = 2.
$$

Similarly,

$$
\lim_{x \to -\infty} \frac{2x^2 + x - 1}{x^2 - 1} = 2.
$$

So $y = 2$ is the only horizontal asymptote. Note that

$$
\frac{2x^2 + x - 1}{x^2 - 1} = 2 + \frac{x + 1}{x^2 - 1}.
$$

So there is no oblique asymptote.

**Information using $y'$:** First using the quotient rule

$$
y' = \frac{(x^2 - 1)(2(2x) + 1) - (2x^2 + x - 1)(2x)}{(x^2 - 1)^2}
= \frac{(x + 1)((x - 1)(4x + 1) - (2x^2 + x - 1)(2x))}{(x - 1)(x + 1)^2}
= \frac{(x + 1)(4x^2 - 3x - 1 - 4x^2 + 2x)}{(x - 1)^2(x + 1)^2}
= \frac{(x + 1)(x - 1)}{(x - 1)^2(x + 1)^2}
= -\frac{1}{(x - 1)^2} \quad \text{as } x = -1 \text{ is not in the domain}
$$

Since $y' < 0$ everywhere in the domain, $y = f(x)$ is a decreasing function. Note that $x = 1$ is not in the domain and so $y = f(x)$ has no critical points.

**Information using $y''$:** We compute $y''$.

$$
y'' = \frac{d}{dx} \left( -\frac{1}{(x - 1)^2} \right) = \frac{2}{(x - 1)^3}.
$$

So we have the following picture:
Note that the function \( f(x) \) has no inflection points. So finally we can put all the information together and get the following graph:

Graph of \( y = f(x) \)

Here is another example.

**Example:** Graph the rational function

\[
y = f(x) = \frac{x^2 - 4}{x - 1}.
\]

First note that the given function is neither odd nor even. So there is no associated symmetry in the graph.

**Asymptotes:**

**Vertical Asymptotes:** The only possibility is when \( x = 1 \).

\[
\lim_{x \to 1^+} \frac{x^2 - 4}{x - 1} = -\infty, \quad \lim_{x \to 1^-} \frac{x^2 - 4}{x - 1} = \infty.
\]
So \( x = 1 \) is a vertical asymptote. (Note that for this to happen we needed only one of the limits to be \( \pm \infty \). But it is useful and necessary to know these limits when sketching the graph.)

**Horizontal Asymptotes:** We compute the limits as \( x \to \pm \infty \).

\[
\lim_{x \to \infty} \frac{x^2 - 4}{x - 1} = \lim_{x \to \infty} \frac{x - \frac{4}{x}}{1 - \frac{1}{x}} = \infty.
\]

Similarly,

\[
\lim_{x \to -\infty} \frac{x^2 - 4}{x - 1} = -\infty.
\]

So there are no horizontal asymptotes. On the other hand note that

\[
\frac{x^2 - 4}{x - 1} = x + 1 - \frac{3}{x - 1}.
\]

So \( y = x + 1 \) is an oblique asymptote.

**Information using \( y' \):** We can use Equation 1 to compute the derivative.

\[
y' = 1 + \frac{3}{2(x - 1)^2} = \frac{x^2 - 2x + 4}{(x - 1)^2}.
\]

Note that the numerator \( x^2 - 2x + 4 = (x - 1)^2 + 3 \) is always greater than zero. Also the denominator is always greater than zero. So the derivative \( y' \) is always positive and hence \( y = f(x) \) is an increasing function. Also since \( x = 1 \) is not in the domain, there are no critical points for the function.

**Information using \( y'' \):** We compute \( y'' \) using the formula for \( y' \).

\[
y'' = \frac{6}{(x - 1)^3}.
\]

So we have the following picture:

<table>
<thead>
<tr>
<th>( y'' )</th>
<th>Concave Up</th>
<th>Concave Down</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y''(0) = 6 &gt; 0 )</td>
<td>+1</td>
<td>( y''(2) = -6 &lt; 0 )</td>
</tr>
</tbody>
</table>

Note that the function \( f(x) \) has no inflection points. So finally we can put all the information together and get the following graph:
Graph of $y = f(x)$