On the other hand, $u = 1 - |x|^2$ solves

$$\begin{cases} -\Delta u = 2n & \text{in } U \\ u = 0 & \text{on } \partial U, \end{cases}$$

so

$$1 - |x|^2 = \int_{B(0,1)} 2nG(x,y)\; dy$$

which implies that

$$\int_{B(0,1)} G(x,y)\; dy = \frac{1-|x|^2}{2n} < \frac{1}{2n} \quad (x \in B(0,1)).$$

Therefore

$$\begin{aligned} |u(x)| &= \left| - \int_{\partial B(0,1)} g(y) \frac{\partial G}{\partial \nu}\; dS(y) + \int_{B(0,1)} f(y) G(x,y)\; dy \right| \\ &\leq \max_{x \in \partial B(0,1)} |g| \left| \int_{\partial B(0,1)} \frac{\partial G}{\partial \nu}\; dS(y) \right| + \max_{x \in \partial B(0,1)} |f| \left| \int_{B(0,1)} G(x,y)\; dy \right| \\ &= C(\max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f|) \end{aligned}$$

for all $x \in B(0,1)$. ∎

Evans Ch.2 #6. Use Poisson's formula for the ball to prove

$$r^{n-2} \frac{r - |x|}{(r+|x|)^{n-1}}\; u(0) \leq u(x) \leq r^{n-2} \frac{r+|x|}{(r-|x|)^{n-1}}\; u(0)$$

whenever u is positive and harmonic in $B^0(0,r)$. This is an explicit form of Harnack's inequality.

***Solution*:** Poisson's formula for the ball gives an explicit smooth solution of Poisson's equation on $B^0(0,r)$. Recall that for $g \in C(\partial B(0,r))$ we take

$$u(x) := \frac{r^2 - |x|^2}{n\alpha(n) r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\; dS(y) \qquad (x \in B^0(0,r)).$$

Furthermore, since u is harmonic we have that

$$u(0) = ⨏_{\partial B(0,r)} g(y)\; dS(y) = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(0,r)} g(y)\; dS(y).$$

Note that $u \geq 0$ implies that $g \geq 0$ since g is assumed to be continuous. Therefore,

$$\begin{aligned}
u(x) &= \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, dS(y) \\
&= r^{n-2} \frac{r^2 - |x|^2}{n\alpha r^{n-1}} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, dS(y) \\
&\leq r^{n-2} \frac{r^2 - |x|^2}{n\alpha r^{n-1}} \int_{\partial B(0,r)} \frac{g(y)}{(r-|x|)^n}\, dS(y) \\
&\quad \text{(Since } r - |x| \leq |x-y| \text{ for all } y \in \partial B(0,r) \text{ and } g \geq 0.) \\
&= r^{n-2} \frac{r^2 - |x|^2}{(r-|x|)^n} \frac{1}{n\alpha r^{n-1}} \int_{\partial B(0,r)} g(y)\, dS(y) \\
&= r^{n-2} \frac{r + |x|}{(r-|x|)^{n-1}} ⨏_{\partial B(0,r)} g(y)\, dS(y) \\
&= r^{n-2} \frac{r + |x|}{(r-|x|)^{n-1}}\, u(0).
\end{aligned}$$

The other inequality is shown analogously by noting that $|x-y| \leq r + |x|$. ∎

Evans Ch.2 #7. Prove *Poisson's formula for the ball*, which states that for $g \in C(\partial B(0,r))$ and for

$$u := \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, dS(y) \qquad (x \in B^0(0,r))$$

we have

(i) $u \in C^\infty(B^0(0,r))$,

(ii) $\Delta u = 0$ in $B^0(0,r)$, and

(iii) $\lim_{x \to x^0,\ x \in B^0(0,r)} u(x) = g(x^0)$ for each point $x^0 \in \partial B(0,r)$.

(Hint: Since $u \equiv 1$ solves

$$\begin{cases} \Delta u = 0 & \text{in } B^0(0,r) \\ u = g & \text{on } \partial B(0,r). \end{cases}$$

for $g \equiv 1$, the theory automatically implies

$$\int_{\partial B(0,1)} K(x,y)\, dS(y) = 1$$

for each $x \in B^0(0,1)$.)

Solution: Recall that $G(x,y) := \Phi(y-x) - \phi^x(y)$ where Φ is the *fundamental solution of Laplace's equation* and ϕ^x is a "corrector function" depending on the domain in question which is harmonic in that region. Therefore $y \mapsto G(x,y)$ is harmonic and since $G(x,y) = G(y,x)$ for $y \neq x$ (a theorem), we have that $x \mapsto G(x,y)$ is also harmonic. Thus $x \mapsto -\frac{\partial G}{\partial \nu} = K(x,y)$ is harmonic for $x \in B^0(0,r)$, $y \in \partial B(0,r)$.

A direct calculation shows that

$$\int_{\partial B(0,1)} K(x,y)\, dS(y) = 1$$

for each $x \in B^0(0,1)$. Since g is continuous on a compact set it is bounded, and consequently u as defined is likewise bounded. Now, since $x \mapsto K(x,y)$ is a harmonic function, it is smooth for $x \neq y$. We easily verify that $u \in C^\infty(B^0(0,r))$, with

$$\Delta u(x) = \int_{\partial B(0,r)} \Delta_x K(x,y)\ g(y)\ dy = 0, \qquad x \in B^0(0,r).$$

Now fix $x^0 \in \partial B(0,r), \epsilon > 0$. Choose $\delta > 0$ small enough to insure that

$$|g(y) - g(x^0)| < \epsilon, \qquad \text{if } |y - x^0| < \delta, \qquad y \in \partial B(0,r).$$

Then if $|x - x^0| < \frac{\delta}{2}, x \in B^0(0,r)$,

$$\begin{aligned} |u(x) - g(x^0)| &= \left| \int_{\partial B(0,r)} K(x,y)[g(y) - g(x^0)]\ dy \right| \\ &\leq \int_{\partial B(0,r) \cap B(x^0,\delta)} K(x,y)|g(y) - g(x^0)|\ dy \\ &\quad + \int_{\partial B(0,r) - B(x^0,\delta)} K(x,y)|g(y) - g(x^0)|\ dy \\ &=: I + J. \end{aligned}$$

Now, since the integral of K over the boundary is equal to 1 (as in the hint), and $|g(y) - g(x^0)| < \epsilon$, we have that

$$I \leq \epsilon \int_{\partial B(0,r)} K(x,y)\ dy = \epsilon.$$

Furthermore if $|x - x^0| \leq \frac{\delta}{2}$ and $|y - x^0| \geq \delta$, we have

$$|y - x^0| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2}|y - x|;$$

and so $|y - x| \geq \frac{1}{2}|y - x^0|$. Thus

$$\begin{aligned} J &\leq 2||g||_{L^\infty} \int_{\partial B(0,r) - B(x^0,\delta)} K(x,y)\ dy \\ &\leq \frac{2||g||_{L^\infty}(r^2 - |x|^2)}{n\alpha(n)r} \int_{\partial B(0,r) - B(x^0,\delta)} \frac{2^n}{|y - x^0|^n}\ dy \\ &= C(r^2 - |x|^2) \\ &\to 0 \qquad \text{as } x \to r. \quad \blacksquare \end{aligned}$$

Evans Ch.2 #8. Let u be the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

given by Poisson's formula for the half-space. Assume that g is bounded and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+, |x| \leq 1$. Show Du is *not* bounded near $x = 0$. (Hint: Estimate $\frac{u(\lambda e_n) - u(0)}{\lambda}$.)

Solution: Recall that the fundamental solution of Laplace's equation with Dirichlet boundary conditions in the upper half-space is

$$u(x) := \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\, dy.$$

Note that since $x = \lambda e_n$ and $y \in \partial\mathbb{R}^n_+$ we have $|x-y| = \sqrt{\lambda^2 + |y|^2}$. Since we have extended u continuously up to the boundary of $\mathbb{R}^n_+$, $u(0) = g(0) = 0$ in this case and

$$\begin{aligned}
\frac{u(\lambda e_n) - u(0)}{\lambda} &= \frac{u(\lambda e_n)}{\lambda} \\
&= \frac{2\lambda}{n\lambda\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{(\lambda^2 + |y|^2)^{n/2}}\, dy \\
&= \frac{2}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{(\lambda^2 + |y|^2)^{n/2}}\, dy \\
&= \frac{2}{n\alpha(n)} \left[\int_{\mathbb{R}^{n-1}\setminus B^{n-1}(0,1)} \frac{g(y)}{(\lambda^2+|y|^2)^n}\, dy + \int_{B^{n-1}(0,1)} \frac{|y|}{(\lambda^2+|y|^2)^n}\, dy \right] \\
&=: \frac{2}{n\alpha(n)}(I+J)
\end{aligned}$$

Now g is a bounded function so let's call this bound C. Then we see that I is a convergent integral since

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}\setminus B^{n-1}(0,1)} \frac{|g(y)|}{(\lambda^2+|y|^2)^{n/2}}\, dy &\leq C \int_{\mathbb{R}^{n-1}\setminus B^{n-1}(0,1)} \frac{1}{(\lambda^2+|y|)^{n/2}}\, dy \\
&= C \int_1^\infty \left[\int_{\partial B^{n-1}(0,1)} \frac{1}{(\lambda^2+|y|^2)^{n/2}}\, dS \right] dr \\
&= C \int_1^\infty \frac{(n-1)\alpha(n-1)r^{n-2}}{(\lambda^2+r^2)^{n/2}}\, dr \\
&= C(n-1)\ \alpha(n-1) \int_1^\infty \frac{r^{n-2}}{(\lambda^2+r^2)^{n/2}}\, dr \\
&< \infty
\end{aligned}$$

by the *integral*, *limit comparison*, and *p-tests* for integrals. Hence we conclude that I is a convergent integral. However,

$$\begin{aligned}
\int_{B^{n-1}(0,1)} \frac{g(y)}{(\lambda^2+|y|^2)^{n/2}}\, dy &= \int_{B^{n-1}(0,1)} \frac{|y|}{(\lambda^2+|y|^2)^{n/2}}\, dy \\
&= \int_0^1 \left[\int_{\partial B^{n-1}(0,r)} \frac{r}{(\lambda^2+r^2)^{n/2}}\, dS \right] dr \\
&= \int_0^1 \frac{r}{(\lambda^2+r^2)^{n/2}} \left[\int_{\partial B^{n-1}(0,r)} dS \right] dr \\
&= \int_0^1 \frac{r}{(\lambda^2+r^2)^{n/2}} \cdot \mathrm{vol}\big(\partial B^{n-1}(0,r)\big)\, dr \\
&= \int_0^1 \frac{r}{(\lambda^2+r^2)^{n/2}} \cdot (n-1)\ \alpha(n-1) r^{n-2}\, dr \\
&= (n-1)\ \alpha(n-1) \int_0^1 \frac{r^{n-1}}{(\lambda^2+r^2)^{n/2}}\, dr \\
&= \infty,
\end{aligned}$$

by the *integral*, *limit comparison*, and *p-tests* for integrals. Hence we conclude that J is a divergent integral for all λ. Therefore,

$$Du \cdot (\lambda e_n) = \lim_{\lambda \to 0} \frac{u(\lambda e_n) - u(0)}{\lambda} = \infty,$$

so Du is unbounded, even though u *is* bounded. ∎

Evans Ch.9 #9. Let U^+ denote the open half-ball $\{x \in \mathbb{R}^n \mid |x| < 1, x_n > 0\}$. Assume $u \in C^2(\bar{U}^+)$ is harmonic in U^+, with $u = 0$ on $\partial U \cup \{x_n = 0\}$. Set

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, \ldots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

for $x \in U = B^0(0,1)$. Prove v is harmonic in U.

***Solution*:** Clearly v is harmonic on $B^0(0,1)\backslash\{x_n = 0\}$ since it is equal to either of the harmonic functions $\pm u$. By the symmetry of v, it's clear that on the set $\{x_n = 0\}$, v satisfies the *Mean Value property*, and is therefore harmonic there as well. ∎

Evans Ch.2 #10. Suppose that u is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.

(i) Show $u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

(ii) Use (i) to show $v(x,t) := x \cdot Du(x,t) + 2tu_t(x,t)$ solves the heat equation as well.

***Solution*:**

(i) We have

$$\begin{aligned} \frac{d}{dt}u_\lambda - \Delta u_\lambda &= \frac{d}{dt}u(\lambda x, \lambda^2 t) - \Delta u(\lambda x, \lambda^2 t) \\ &= \lambda^2 u_t(\lambda x, \lambda^2 t) - \lambda^2 \Delta u_t(\lambda x, \lambda^2 t) \\ &= \lambda^2 \big(u_t(y,z) - \Delta u(y,z)\big) \\ &\quad \text{(after the change of variables } y = \lambda x, z = \lambda^2 t\text{)} \\ &= 0. \end{aligned}$$

(ii) We can show that v solves the heat equation without using (i) by proceeding via direct calculation. We introduce the *heat operator* $\Delta_h := \frac{\partial}{\partial t} - \Delta$ to abbreviate notation. First note that if $\Delta_h u = 0$ then $\Delta_h u_{x_i} = \Delta_h u_t = 0$ as well, a fact relying on the equality of mixed partial derivatives:

$$\begin{aligned} \Delta_h u_t &= (u_t)_t - \Delta u_t \\ &= (u_t)_t - \sum_{i=1}^n u_{tx_ix_i} \\ &= (u_t)_t - \sum_{i=1}^n u_{x_ix_it} \\ &= (u_t)_t - \left(\sum_{i=1}^n u_{x_ix_i}\right)_t \\ &= (u_t - \Delta u)_t \\ &= (0)_t \\ &= 0. \end{aligned}$$