1) Solitons and Bores

Under appropriate assumptions, the shallow water equations can be simplified to the Korteweg-de Vries-Burgers equation,

\[ u_t + uu_x + \delta u_{xxx} - \epsilon u_{xx} = 0, \]

which describes the displacement of a free surface of water from its equilibrium position, where \( \delta > 0 \) represents the strength of dispersive effects and \( \epsilon \geq 0 \) represents dissipation. The quantity \( u(x, t) \) is essentially \( h(x, t) - h_0 \) in the shallow water equations. In the previous exercise we saw how the slope of an upstream wave could blow-up in finite time, and we commented that dissipation might limit this effect, leading to a balance which enables a waveform (Bore) to smoothly connect two different water levels.

The goal of this exercise is to find this solution of (1). To this end we look for special traveling wave solutions which depend on \( x \) and \( t \) only through a traveling coordinate \( \eta = x - ct \) for some speed \( c > 0 \). That is a solution \( u \) of the form

\[ u(x, t) = \phi(x - ct) = \phi(\eta), \]

for some function \( \phi \) of a single variable.

(a) Plug \( \phi(\eta) \) into (1), use the relations \( \eta_x = 1 \) and \( \eta_t = -c \) to reduce the equation to a second order ODE for \( \phi \). The ODE can be integrated once exactly, you should obtain

\[ \delta \phi'' - \epsilon \phi' + \frac{\phi^2}{2} - c\phi = A, \]

where \( A \) is an unknown constant of integration. To find Bore solutions we anticipate that \( \lim_{\eta \to \infty} \phi(\eta) = 0 \) as well as \( \lim_{\eta \to \infty} \phi'(\eta) = 0 \). Argue that consequently \( A = 0 \).

(b) Making the substitution \( v = \phi \) and \( w = \phi' \), write the second order ODE as a first order system, find the fixed points (one at \( (0, 0) \)), linearize about both fixed points, and show that for \( \epsilon > 0 \) the fixed point at the origin is a saddle while the other is either an unstable node or an unstable spiral depending on the size of \( \frac{\epsilon^2}{\delta c} \).

(c) The famous KdV equation has \( \epsilon = 0 \) above. Since it has no dissipation it is sensible to look for a conserved quantity \( E(v, w) \). Find it. Show that the origin is still a saddle, but
that the other fixed point is a nonlinear center. Take $\delta = 1$ and $c = 1$ and plot the phase diagram, in particular use the nullclines and find the eigenvectors of the linearization about the origin and use the explicit formula for the level set of $E$ passing through the origin to plot the stable and unstable manifolds of the origin lying in the half-plane $v \geq 0$. Show that they form a homoclinic orbit. Show that the other fixed point is a local maximum of $E$. Finally, give a rough plot of the $v$ component of the homoclinic orbit as a function of $\eta$ (essentially our time variable). This graph is the soliton solution of the KdV equations.

(d) (The Boring part) Take $\epsilon > 0$ and show that the conserved quantity $E$ of part $c$ is now a bit leaky. Particularly show that $\frac{dE(v(\eta), w(\eta))}{d\eta} \leq 0$. Take $\delta = \epsilon = 1$ and $c = 2$ and plot the phase diagram, paying particular attention to the nullclines and use the Liapunov functional $E$ to argue that the stable and unstable manifolds of the origin can no-longer intersect, and explain why the stable manifold must spiral into the other fixed point in backwards time (this is now a heteroclinic orbit and the breaking of the homoclinic orbit is termed a homoclinic bifurcation). Plot the $v$ component of the heteroclinic orbit a function of $\eta$, in particular note that $v \to 0$ as $\eta \to \infty$ while $v$ tends to a nonzero limit as $\eta \to -\infty$, (to help give a reasonable graph note that $v$ spends lots of time near the fixed points and moves relatively quickly in the interim). This graph is the oscillatory bore solution of the KdVB equation which connects two different water levels, all other solutions tend to $\infty$ as $\eta \to \infty$.

(e)(The monotone Bore) Take $\epsilon = 2$, $\delta = 1$, and $c = 1$. Plot the phase diagram, observe that the other fixed-point is no longer a spiral, find a cute little trapping region which shows that the monotone bore does not oscillate, (i.e. $v(\eta)$ is monotone increasing) and graph the stable manifold of the origin, showing that it is still a heteroclinic orbit. Again, plot $v$ vs $\eta$ and explain why this is called the monotone bore.