Non-book Exercises

For the following exercises, the $l^p$ norm is defined on $\mathbb{R}^n$ by

$$
\|\vec{x}\|_p \equiv \left( \sum_{k=1}^{n} |x(k)|^p \right)^{\frac{1}{p}}.
$$

1) Prove for $n = 2$ that $\frac{1}{\sqrt{2}} \|\vec{x}\|_1 \leq \|\vec{x}\|_2$.

2) **Honors Problem** For this problem let $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Such $p$ and $q$ are called conjugate exponents.

(a) Prove Young’s inequality:

$$
|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q},
$$

for all conjugate exponents $p, q$ and all $a, b \in \mathbb{R}$.

*Hint: Compare the area of the box bounded by the lines $x = 0$, $y = 0$, $x = a$, and $y = b$ to the area between the $x$ axis and the curve $y = x^{q-1}$ and between the $y$ axis and the curve $x = y^{p-1}$.* Use the fact that $p, q$ conjugate exponents implies $(p-1)(q-1) = 1$, so the two curves are the same.

(b) Use Young’s inequality to prove Hölder’s inequality:

$$
\sum_{k=1}^{n} |x(k)y(k)| \leq \|\vec{x}\|_p \|\vec{y}\|_q,
$$

for all conjugate exponents $p$ and $q$.

(c) Use Hölder’s inequality to prove that

$$
\frac{1}{n^{1/q}} \|\vec{x}\|_1 \leq \|\vec{x}\|_p,
$$
for all $\vec{x} \in \mathbb{R}^n$ and all conjugate exponents $p$ and $q$.

(d) Use Hölder’s inequality to prove the triangle inequality for the $l^p$ norm, $p \geq 1$. That is

$$\|\vec{x} + \vec{y}\|_p \leq \|\vec{x}\|_p + \|\vec{y}\|_p,$$

for all $\vec{x}, \vec{y} \in \mathbb{R}^n$. 