Math 869:Assignment 2

Due Friday February 28

Problem 1. Construct a simply connected covering space of the space $X \subset \mathbf{R^3}$ that is the union of a 2-sphere and a diameter of the 3-ball.

Problem 2. Show that if a path-connected, locally path-connected space X has finite $\pi_1(X)$, then every map $X \to S^1$ is nullhomotopic. [*Hint:* Use the covering space $\mathbf{R} \to S^1$.]

Problem 3. Let a, b be the generators of $\pi_1(S^1 V S^1)$ corresponding to the two S^1 summands. Draw a picture of the covering space of $S^1 V S^1$ corresponding to the normal subgroup generated by a^2, b^2 and $(ab)^4$, and PROVE that this covering space is indeed the correct one.

Problem 4. Given a group and a normal subgroup N, show that there exists a normal covering space $\tilde{X} \to X$ with $\pi_1(X) \simeq G$, $\pi_1(\tilde{X}) \simeq N$, and deck transformation group $\mathrm{Aut}(\tilde{X}) \simeq G/N$. [*Hint:* You may find Corollary 1.28 of Hatcher's book useful.]

Problem 5. For a path-connected, locally path-connected and semi locally simply connected space X, call a path-connected covering space $\tilde{X} \to X$ ABELIAN if it is normal (regular) and has abelian deck transformation group. Show that X has an abelian covering space that is a covering space of any other abelian covering space of X and that such a "universal" abelian covering space is unique up to isomorphism. Describe this universal abelian cover explicitly for $X := S^1 V S^1$.

Problem 6. Solve Exercise 20 on page 81 of Hatcher's book.

Problem 7. Consider the linear transformation $\tau : \mathbf{R}^2 \to \mathbf{R}^2$, defined by $\tau(x,\ y) = (2x, y/2)$. It generates an action of $\mathbf{Z}$ on $X := \mathbf{R}^2 - \{0\}$. Show that this action is a covering space action and calculate the fundamental group $\pi_1(X/\mathbf{Z}$. Is the quotient space a Hausdorff space? Justify your answer.

Problem 8. Solve Exercise 7 on page 87 of Hatcher's book.