Math 869: Final Assignment

Due Thursday May 1 by 12:00pm

Please return to my office: D-323 Wells Hall.

Problem 1. Let $X_n \subset \mathbf{R}^3$ denote the union of n lines through the origin. Compute the fundamental group $\pi_1(\mathbf{R}^3 \setminus X_n)$. Use appropriate theorems to justify your computation and show your work.

Problem 2. Construct a 2-dimensional CW complex with fundamental group $G \approx < a, b, c \mid a^2 b = 1, c^3 = 1 >$.

Problem 3. Find all the connected covering spaces of the topological space X that is the wedge of two real projective planes. That is $X = \mathbf{R}P^2 \vee \mathbf{R}P^2$. Which of these covering spaces are normal?

Problem 4. a) Prove that the spaces $X := S^1 \times S^1$ and $Y := S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions.

b) Prove that the universal coverings of X and Y do NOT have isomorphic homology groups in all dimensions.

Problem 5. Compute the homology groups of the 2-complex $X(m,n)$ obtained as quotient space of $S^1 \times S^1$ after the following identification: Identify points in the circle $S^1 \times \{x_0\}$ that differ by $\frac{2\pi}{m}$ rotation and identify points on the circle $\{x_0\} \times S^1$ that differ by $\frac{2\pi}{n}$ rotation.

Problem 6. For a topological space X and a subspace $A \subset X$ consider the short exact sequence of chain complexes

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X,A) \to 0,$$

where the homomorphism i is induced by the inclusion $A \hookrightarrow X$ and j is the quotient map $C_n(X) \to C_n(X,A) := \frac{C_n(X)}{C_n(A)}$. This short exact sequence above is split and thus we get $C_n(X) \approx C_n(A) \oplus C_n(X,A)$. (*You don't need to prove that.*)

Does the splitting above always yield splittings $H_n(X) \approx H_n(A) \oplus H_n(X,A)$? If your answer is YES, prove your claim. If your answer is NO then explain why not and give an example to justify it.

Problem 7. Compute the cohomology groups $H^i(\mathbf{R}P^3)$ and the cohomology groups $H^i(\mathbf{R}P^2 \vee S^3)$, for all $i \geq 0$.

Can we use cup products to distinguish $\mathbf{R}P^3$ and $\mathbf{R}P^2 \vee S^3$?