Geometric structures of 3-manifolds and quantum invariants

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Settings and general theme of talk

**3-manifolds:** $M$ = compact, orientable, with empty or tori boundary.

**Knots:** Smooth embedding $K : S^1 \rightarrow S^3$. Knots $K_1, K_2$ are equivalent if $f(K_1) = K_2$, $f$ orientation preserving diffeomorphism of $S^3$.

**Talk:** Relations among three perspectives.

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**Combinatorial presentations**
- knot diagrams, triangulations
- Cut/paste

**3-manifold topology/geometry**
- Geometric structures on $M$ (e.g. $M = S^3 \setminus K$) and geometric invariants

**Physics originated invariants**
- Quantum invariants of knots/3-manifolds
Warm up: 2-d Model Geometries:

For this talk, an \( n \)-dimensional *model geometry* is a simply connected \( n \)-manifold with a “homogeneous” Riemannian metric. In dimension 2, there are exactly three model geometries, up to scaling:

- **Spherical**
  - Curvature: +1
  - Area \( T \) = \((\alpha + \beta + \gamma) - \pi\)

- **Euclidean**
  - Curvature: 0
  - Area \( T \) = \( \pi \)
  - \( \alpha + \beta + \gamma = \pi \)

- **Hyperbolic**
  - Curvature: -1
  - Area \( T \) = \( \pi - (\alpha + \beta + \gamma) \)
Geometrization (a.k.a. Uniformization) in 2-d:

Every (closed, orientable) surface can be written as $S = X/G$, where $X$ is a model geometry and $G$ is a discrete group of isometries.

\[ X = S^2 \quad X = \mathbb{E}^2 \quad X = \mathbb{H}^2 \]
Geometrization (a.k.a. Uniformization) in 2-d:

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$X = S^2$  $X = \mathbb{E}^2$  $X = \mathbb{H}^2$

Geometry relates to topology: $k \cdot \text{Area}(S) = 2\pi \chi(S)$, $k = 1, 0, -1$ (curvature).
Geometrization in 3-d:

In dimension 3, there are eight model geometries:

\[ X = S^3, E^3, H^3, S^2 \times \mathbb{R}, H^2 \times \mathbb{R}, Sol, Nil, \widetilde{SL}_2(\mathbb{R}) \]

**Theorem (Thurston 1980 + Perelman 2003)**

For every (compact, oriented) 3-manifold \( M \), there is a **canonical** way to cut \( M \) along spheres and tori into pieces \( M_1, \ldots, M_n \), such that each piece is \( M_i = X_i / G_i \), where \( G_i \) is a discrete group of isometries of the model geometry \( X_i \).

- **Canonical**: “Unique” collection of spheres and tori.
- The Poincare conjecture is a special case (\( S^3 \) is the only compact model).
- Hyperbolic 3-manifolds are a prevalent, rich and very interesting class.
- Because of cutting along tori, manifolds with toroidal boundary will naturally arise. Knot complements fit in this class.
Given $K$ remove an open tube around $K$ to obtain the Knot complement: Notation. $M_K = S^3 \setminus K$.
Knots complements; nice 3-manifolds with boundary:

Given $K$ remove an open tube around $K$ to obtain the *Knot complement*:

**Notation.** $M_K = S^3 \setminus K$.

Knot complements can be visualized!
More on the Geometric decomposition

Theorem (Knesser, Milnor 60’s, Jaco-Shalen, Johanson 1970, Thurston 1980 + Perelman 2003)

M=oriented, compact, with empty or toroidal boundary.

1. There is a unique collection of 2-spheres that decompose M

\[ M = M_1 \# M_2 \# \ldots \# M_p \# (\# S^2 \times S^1)^k, \]

where \( M_1, \ldots, M_p \) are compact orientable irreducible 3-manifolds.

2. For M=irreducible, there is a unique collection of disjointly embedded essential tori \( T \) such that all the connected components of the manifold obtained by cutting M along \( T \), are either Seifert fibered manifolds or hyperbolic.

- **Seifert fibered manifolds**: For this talk, think of it as \( S^1 \times \) surface with boundary + union of solid tori.

- **Hyperbolic**: Interior admits complete, hyperbolic metric of finite volume.
Thee types of knots:

**Satellite Knots:** Complement contains embedded “essential” tori; There is a canonical (finite) collection of such tori.

**Torus knots:** Knot embeds on standard torus in $T$ in $S^3$ and is determined by its class in $H_1(T)$. Complement is SFM.

**Hyperbolic knots:** Rest of them.
Rigidity for hyperbolic 3-manifolds:

**Theorem (Mostow, Prasad 1973)**

Suppose $M$ is compact, oriented, and $\partial M$ is a possibly empty union of tori. If $M$ is hyperbolic (that is: $M \setminus \partial M = \mathbb{H}^3 / G$), then $G$ is unique up to conjugation by hyperbolic isometries. In other words, a hyperbolic metric on $M$ is essentially unique.

$M =$ hyperbolic 3-manifold:
- By rigidity, every geometric measurement of $M$ (volume, areas of surface etc.) is a *topological invariant*
- In practice $M$ is represented by combinatorial data such as, a *triangulation*, a *Heegaard diagram*, a *Dehn surgery diagram* or a *knot diagram* (in case of knot complements).

**Challenging Question:** How do we see geometry in the combinatorial descriptions of $M$? Can we calculate/estimate geometric invariants from combinatorial ones? *(Highly active research area)*
Recall $M$ uniquely decomposes along spheres and tori into disjoint unions of Seifert fibered spaces and hyperbolic pieces $M = S \cup H$.

Gromov, Thurston, 80’s:

**Gromov norm of $M$:** $||M|| = v_3 \operatorname{Vol}(H)$, $\operatorname{Vol}(H)$ is the sum of the hyperbolic volumes of components of $H$ and $v_3$ is the volume of the regular hyperbolic tetrahedron.

$||M||$ is additive under disjoint union and connected sums of manifolds.

If $M$ hyperbolic $||M|| = v_3 \operatorname{Vol}(M)$.

If $M$ Seifert fibered then $||M|| = 0$

If $M$ contains an embedded torus $T$ and $M'$ is obtained from $M$ by cutting along $T$ then

$$||M|| \leq ||M'||.$$

Moreover, the inequality is an equality if $T$ is incompressible in $M$. 
Quantum invariants: Jones Polynomials

1980’s: Ideas originated in physics and in representation theory led to vast families invariants of knots and 3-manifolds. (*Quantum invariants*)

- **Jones Polynomials**: Discovered by V. Jones (1980’s); using braid group representations coming from the theory of certain operator algebras (sub factors).
- Can be calculated from any link diagram using, for example, Kaufman states:
- Two choices for each crossing, A or B resolution.

Choice of A or B resolutions for all crossings: *state* $\sigma$.
- Assign a “weight” to every state.
- JP calculated as a certain “state sum” over all states of any diagram.
Quantum invariants: Colored Jones Polynomials

For this talk we discuss:

- The *Colored Jones Polynomials*: Infinite sequence of Laurent polynomials \( \{J_{K,n}(t)\}_n \) encoding the *Jones polynomial* of \( K \) and these of the links \( K^s \) that are the *parallels* of \( K \).

- Formulae for \( J_{K,n}(t) \) come from *representation theory of Lie Groups!*: representation theory of \( SU(2) \) (decomposition of tensor products of representations). For example, they look like:

  \[
  J_{K,1}(t) = 1, \quad J_{K,2}(t) = J_K(t) - \text{Original JP}, \\
  J_{K,3}(t) = J_{K^2}(t) - 1, \quad J_{K,4}(t) = J_{K^3}(t) - 2J_K(t), \ldots
  \]

- \( J_{K,n}(t) \) can be calculated from any knot diagram via processes such as *Skein Theory, State sums, R-matrices, Fusion rules*....
**The CJP predicts Volume?**

**Question:** How do the **CJP** relate to geometry/topology of knot complements?

Kashaev+ H. Murakami - J. Murakami (2000) proposed

**Volume conjecture.** Suppose $K$ is a knot in $S^3$. Then

$$2\pi \cdot \lim_{n \to \infty} \frac{\log |J_K(e^{2\pi i/n})|}{n} = v_3 ||S^3 \setminus K||$$

- The conjecture is wide open: Few verifications by brute force calculations.
- Knots up to 7 crossings Ohtsuki),
- Simple families of knots of zero Gromov norm zero (Zheng, Kashaev).

**Some difficulties:**
- “State sum” for $J_K(e^{\pi i/2n})$ very oscillating; is often $J_K(e^{\pi i/2n}) = 0$.
- No good behavior of $J_K(e^{\pi i/2n})$ with respect to geometric decompositions.
Coarse relations: Colored Jones polynomial

For a knot $K$, and $n = 1, 2, \ldots$, we write its \textit{n-colored Jones polynomial}:

$$J_{K,n}(t) := \alpha_n t^{m_n} + \beta_n t^{m_n-1} + \cdots + \beta'_n t^{k_n+1} + \alpha'_n t^{k_n} \in \mathbb{Z}[t, t^{-1}]$$

- (Garoufalidis-Le, 04): Each of $\alpha'_n, \beta'_n \ldots$ satisfies a \textit{linear recursive relation} in $n$, with integer coefficients.
  
  (e.g. $\alpha'_{n+1} + (-1)^n \alpha'_n = 0$).

- Given a knot $K$ any diagram $D(K)$, there exist \textit{explicitly given} functions $M(n, D)$ $m_n \leq M(n, D)$. For \textit{nice} knots where $m_n = M(n, D)$ we have \textit{stable coefficients}.

- (Dasbach-Lin, Armond) If $m_n = M(n, D)$, then
  $$\beta'_K := |\beta'_n| = |\beta'_2|,$$
  and
  $$\beta_K := |\beta_n| = |\beta_2|,$$
  for every $n > 1$.

- Stable coefficients control the volume of the link complement.
A Coarse Volume Conjecture

**Theorem (Dasbach-Lin, Futer-K.-Purcell, Giambrone, 05-’15’)**

There universal constants $A, B > 0$ such that for any hyperbolic link that is nice we have

$$A(\beta'_K + \beta_K) \leq \text{Vol}(S^3 \setminus K) < B(\beta'_K + \beta_K).$$

**Question.** Does there exist function $B(K)$ of the coefficients of the colored Jones polynomials of a knot $K$, that is easy to calculate from a “nice” knot diagram such that for hyperbolic knots, $B(K)$ is coarsely related to hyperbolic volume $\text{Vol}(S^3 \setminus K)$?

Are there constants $C_1 \geq 1$ and $C_2 \geq 0$ such that

$$C_1^{-1}B(K) - C_2 \leq \text{Vol}(S^3 \setminus K) \leq C_1B(K) + C_2,$$

for all hyperbolic $K$?

- C. Lee, Proved CVC for more classes of knots (2017)
Families of real valued invariants $TV_r(M, q)$ of a compact oriented 3-manifold $M$; indexed by a positive integer $r$, the level and for each $r$ they depend on an $2r$-th root of unity, $q$. [Turaev-Viro, 1990]

$TV_r(M, q)$ are combinatorially defined invariants and can be computed from triangulations of $M$ by a state sum formula. Sums involve quantum $6j$-sympols. Terms are highly “oscillating”. Combinatorics relay have roots on representations of Lie groups.

For this talk: $TV_r(M) := TV_r(M, e^{\frac{2\pi i}{r}})$, $r = odd$ and $q = e^{\frac{2\pi i}{r}}$.

For experts: These correspond to the $SO(3)$ quantum group.

(Q. Chen- T. Yang, 2015): compelling experimental evidence supporting

Conjecture : For $M$ compact, orientable

$$\lim_{r \to \infty} \frac{2\pi}{r} \log (TV_r(M, e^{\frac{2\pi i}{r}})) = v_3 ||M||,$$

where $r$ runs over odd integers.
Recent results (Detcherry-K.-Yang, 2016)

- For $M = S^3 \setminus L$, a link complement in $S^3$, the invariants $TV_r(M)$ can be expressed in terms of the colored Jones polynomial of $L$.
- Gave first examples of “large $r$” asymptotics of $TV_r(M, e^{2\pi i r})$ are calculated and verified the Chen-Yang conjecture for some link complements (Borromean rings, Figure-eight-hyperbolic manifolds).
- Proved Conjecture for Knots of Gromov norm zero.
- Conjecture is compatible with disjoint unions of links and connect sums (Warning: Original volume conjecture is not!).
- We have
  $$\liminf_{r \to \infty} \frac{2\pi}{r} \log( TV_r(M) ) \geq 0.$$ 
- We discover “new” exponential growth phenomena of the colored Jones polynomial at values that are not predicted by the Kashaev-Murakami-Murakami conjecture or generalizations.
Recent results, con’t

$$LTV(M) = \limsup_{r \to \infty} \frac{2\pi}{r} \log(TV_r(M))$$

where $r$ runs over all odd integers. The main result of this article is the following:

**Theorem (Detcherry-K., 2017)**

*There exists a universal constant $B > 0$ such that for any compact orientable 3-manifold $M$ with empty or toroidal boundary we have*

$$B \cdot LTV(M) \leq ||M||,$$

*where the constant $B$ is about $1.1964 \times 10^{-10}$.*

**Corollary**

*For any link $K \subset S^3$ with $||S^3 \setminus K|| = 0$, we have*

$$LTV(M) = \lim_{r \to \infty} \frac{2\pi}{r} \log(TV_r(M \setminus K)) = v_3 ||S^3 \setminus K|| = 0,$$
Why are TV invariants “better”? 

- TV invariants are defined for all compact, oriented 3-manifolds.
- TV invariants are defined on triangulations of 3-manifolds: For hyperbolic 3-manifolds the (hyperbolic) volume can be estimated/calculated from appropriate triangulations.
- TV invariants are part of a Topological Quantum Field Theory (TQFT) and they can be computed by cutting and gluing 3-manifolds along surfaces. The TQFT behaves well when cutting along spheres and tori; in particular with respect to prime and JSJ decompositions.
- For experts: The TQFT is the $SO(3)$- Reshetikhin-Turaev and Witten TQFT as constructed by Blanchet, Habegger, Masbaum and Vogel (1995)
Outline of proof of main result:

1. Study the large-r asymptotic behavior of the quantum $6j$-symbols, and using the state sum formulae for the invariants $TV_r$, to prove give linear upper bound of $LTV(M)$ in terms of the number of tetrahedra in any triangulation of $M$. In particular, $LTV(M) < \infty$.

2. Use step (1) and a theorem of Thurston to show that there is $C > 0$ such that for any hyperbolic 3-manifold $M$, $LTV(M) \leq C||M||$.

3. Use TQFT properties to show that if $M$ is a Seifert fibered manifold, then $LTV(M) = ||M|| = 0$.

4. Show that if $M$ contains an embedded tori $T$ and $M'$ is obtained from $M$ by cutting along $T$ then

\[ LTV(M) \leq LTV(M'). \]

5. Show that $LTV(M)$ is (sub)additive under connected sum and disjoint unions.

6. Use geometric decomposition of 3-manifolds and parallel behavior of $LTV(M)$ and $||M||$ to prove theorem.
New exponential growth results:

- Chen-Tian Conjecture implies that

\[ ITV(M) := \liminf_{r \to \infty} \frac{2\pi}{r} \log(TV_r(M)) > 0, \]

for any complete, hyperbolic 3-manifold of finite volume. This property is very hard to establish using the state sum expressions of the Tureav-Viro invariants.

- Detcherry-K. showed that for \( M, M' \) compact orientable with empty or toroidal boundary, and such that \( M \) is obtained by Dehn filling from \( M' \) we have \( ITV(M') > ITV(M) \). Thus exponential growth of the Turaev-Viro invariants for \( M \) implies the exponential growth for the invariants of \( M' \).

- We have

**Corollary**

Let \( L \) be a link in \( S^3 \) that contains the figure-8 knot or the Borromean rings as a sublink. Then we have

\[ ITV(S^3 \setminus L) \geq 2v_3. \]