Topology in the degree of the colored Jones Polynomial

Effie Kalfagianni
Michigan State University

April 24, 2015
The Jones polynomial was discovered by Jones in 1984.
Garoufalidis and Le proved that, for a fixed knot $K$ the colored Jones function $J_K(n)$ satisfies a non-trivial linear recurrence relation of the form

$$a_d(t^{2n}, t)J_K(n + d) + \cdots + a_0(t^{2n}, t)J_K(n) = 0$$

for all $n$, where $a_j(u, v) \in \mathbb{C}[u, v]$. 
Garoufalidis and Le proved that, for a fixed knot $K$ the colored Jones function $J_K(n)$ satisfies a non-trivial linear recurrence relation of the form

$$a_d(t^{2n}, t)J_K(n + d) + \cdots + a_0(t^{2n}, t)J_K(n) = 0$$

for all $n$, where $a_j(u, v) \in \mathbb{C}[u, v]$.

Example: for the trefoil the colored Jones polynomial is

$$J_K(n) = t^{-6(n^2-1)} \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{24j^2+12j} \frac{t^{8j+2} - t^{-(8j+2)}}{t^2 - t^{-2}}.$$
Garoufalidis and Le proved that, for a fixed knot $K$ the colored Jones function $J_K(n)$ satisfies a non-trivial linear recurrence relation of the form

$$a_d(t^{2n}, t)J_K(n + d) + \cdots + a_0(t^{2n}, t)J_K(n) = 0$$

for all $n$, where $a_j(u, v) \in \mathbb{C}[u, v]$.

Example: for the trefoil the colored Jones polynomial is

$$J_K(n) = t^{-6(n^2-1)} \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{24j^2+12j} \frac{t^{8j+2} - t^{-(8j+2)}}{t^2 - t^{-2}}.$$

It satisfies the following linear recurrence relation

$$(t^{8n+12} - 1)J_K(n+2) + (t^{-4n-6} - t^{-12n-10} - t^{8n+10} + t^{-2})J_K(n+1) + (t^{-4n+4} - t^{-12n-8})J_K(n) = 0.$$
\((Mf)(n) = t^{2n}f(n),\quad (Lf)(n) = f(n + 1)\).
The slope conjecture

For a knot $K \subset S^3$, let $N_K$ be a tubular neighborhood of $K$ and let $M_K = S^3 \setminus N_K$ the complement of $K$. Let $\langle \mu, \lambda \rangle$ be the canonical meridian–longitude basis of $H_1(\partial N_K)$. 

An element $p/q \in \mathbb{Q} \cup \{1/0\}$ is called a boundary slope of $K$ if there is an essential surface $(S, \partial S) \subset (M_K, \partial N_K)$, such that $\partial S$ represents $p\mu + q\lambda \in H_1(\partial N_K)$.

Hatcher proved that every knot $K \subset S^3$ has finitely many boundary slopes. We use $bs_K$ to denote the set of boundary slopes of $K$. 

E. Kalfagianni
The slope conjecture

For a knot $K \subset S^3$, let $N_K$ be a tubular neighborhood of $K$ and let $M_K = S^3 \setminus N_K$ the complement of $K$. Let $\langle \mu, \lambda \rangle$ be the canonical meridian–longitude basis of $H_1(\partial N_K)$.

An element $p/q \in \mathbb{Q} \cup \{1/0\}$ is called a boundary slope of $K$ if there is an essential surface $(S, \partial S) \subset (M_K, \partial N_K)$, such that $\partial S$ represents $p\mu + q\lambda \in H_1(\partial N_K)$.
For a knot $K \subset S^3$, let $N_K$ be a tubular neighborhood of $K$ and let $M_K = S^3 \setminus N_K$ the complement of $K$. Let $\langle \mu, \lambda \rangle$ be the canonical meridian–longitude basis of $H_1(\partial N_K)$.

An element $p/q \in \mathbb{Q} \cup \{1/0\}$ is called a *boundary slope* of $K$ if there is an essential surface $(S, \partial S) \subset (M_K, \partial N_K)$, such that $\partial S$ represents $p\mu + q\lambda \in H_1(\partial N_K)$.

Hatcher proved that every knot $K \subset S^3$ has finitely many boundary slopes. We use $bs_K$ to denote the set of boundary slopes of $K$. 

E. Kalfagianni
The slope conjecture

Let \( d_+ [J_K(n)] \) denote the highest degree of \( J_K(n) \) in \( t \), and let \( d_- [J_K(n)] \) denote the lowest degree. Elements of the sets

\[
js_K := \left\{ n^{-2} d_+ [J_K(n)] \right\}' \quad \text{and} \quad js^*_K := \left\{ n^{-2} d_- [J_K(n)] \right\}'
\]

are called \textit{Jones slopes} of \( K \). Here \( \{x_n\}' \) denote the set of its cluster points of the sequence \( \{x_n\} \).
The slope conjecture

Let $d_+[J_K(n)]$ denote the highest degree of $J_K(n)$ in $t$, and let $d_- [J_K(n)]$ denote the lowest degree. Elements of the sets

$$ js_K := \left\{ n^{-2} d_+ [J_K(n)] \right\}' \quad \text{and} \quad js^*_K := \left\{ n^{-2} d_- [J_K(n)] \right\}' $$

are called Jones slopes of $K$. Here $\{x_n\}'$ denote the set of its cluster points of the sequence $\{x_n\}$.

**Slope Conjecture.** For every knot $K \subset S^3$ we have

$$ (js_K \cup js^*_K) \subset bs_K. $$
Example: for the $(-2, 3, 7)$-pretzel knot we have

\[
\begin{align*}
    d_+[J_K(n)] &= 37n^2/2 + 34n + e(n), \\
    d_-[J_K(n)] &= 5n,
\end{align*}
\]

where $e(n)$ is a periodic sequence of period 4.
The slope conjecture

Example: for the \((-2, 3, 7)\)-pretzel knot we have

\[
\begin{align*}
d_+ [ J_K(n) ] &= 37n^2/2 + 34n + e(n), \\
d_- [ J_K(n) ] &= 5n,
\end{align*}
\]

where \(e(n)\) is a periodic sequence of period 4.

The \((-2, 3, 7)\)-pretzel knot is a Montesinos knot and its boundary slopes are given by

\[\{0, 16, 37/2, 20\}.\]
The slope conjecture

Example: for the \((-2, 3, 7)\)-pretzel knot we have

\[
\begin{align*}
d_+ [J_K(n)] &= 37n^2/2 + 34n + e(n), \\
d_- [J_K(n)] &= 5n,
\end{align*}
\]

where \(e(n)\) is a periodic sequence of period 4.

The \((-2, 3, 7)\)-pretzel knot is a Montesinos knot and its boundary slopes are given by

\(\{0, 16, 37/2, 20\}\).

The slope conjecture was confirmed for the following knots:

- alternating knots, knots with up to nine crossings, torus knots, \((-2, 3, 2n + 1)\)-pretzel knots (by Garoufalidis),
- adequate knots (by Futer, Kalfagianni and Purcell),
- 2-fusion knots (by Garoufalidis-Dunfield-Van der Veen).
- “Most” of \((p, q, r)\)-pretzel knots (Lee- Van der Veen)
The slope conjecture

Example: for the \((-2, 3, 7)\)-pretzel knot we have

\[
\begin{align*}
d_+[J_K(n)] &= 37n^2/2 + 34n + e(n), \\
d_-[J_K(n)] &= 5n,
\end{align*}
\]

where \(e(n)\) is a periodic sequence of period 4.

The \((-2, 3, 7)\)-pretzel knot is a Montesinos knot and its boundary slopes are given by

\[\{0, 16, 37/2, 20\}\].

The slope conjecture was confirmed for the following knots:

- alternating knots, knots with up to nine crossings, torus knots, \((-2, 3, 2n + 1)\)-pretzel knots (by Garoufalidis),
- adequate knots (by Futer, Kalfagianni and Purcell),
- 2-fusion knots (by Garoufalidis-Dunfield-Van der Veen).
- “Most” of \((p, q, r)\)-pretzel knots (Lee-Van der Veen)

Remark: Curtis and Taylor were one of the first authors to study the relation between boundary slopes and the degree of the Jones polynomial.
Suppose $K$ is a knot, and $p, q$ are co-prime integers. The $(p, q)$-cable $K_{p,q}$ of $K$ is the satellite of $K$ with pattern $(p, q)$-torus knot.
Suppose $K$ is a knot, and $p, q$ are co-prime integers. The $(p, q)$-cable $K_{p,q}$ of $K$ is the satellite of $K$ with pattern $(p, q)$-torus knot.

v.d. Veen proved that, for $n > 0$ we have

$$J_{K_{p,q}}(n) = t^{-pq(n^2-1)/4} \sum_{k \in S_n} t^{4pk(qk+1)} J_K(2qk + 1)$$

where $S_n$ be the set of all $k$ such that

$$|k| \leq (n - 1)/2 \quad \text{and} \quad k \in \begin{cases} \mathbb{Z} & \text{if } n \text{ is odd,} \\ \mathbb{Z} + 1/2 & \text{if } n \text{ is even.} \end{cases}$$
The slope conjecture for cable knots

The following result provides a relation between the boundary slopes of a knot and those of the cables of the knot.

**Theorem A** (Motegi-Takata; K-Tran) For every knot $K \subset S^3$ and $(p, q)$ co-prime integers we have

$$(q^2 bs_K \cup \{pq\}) \subset bs_{K_{p,q}}.$$
The slope conjecture for cable knots

- The following result provides a relation between the boundary slopes of a knot and those of the cables of the knot.

**Theorem A** (Motegi-Takata; K-Tran) For every knot $K \subset S^3$ and $(p, q)$ co-prime integers we have

\[
(q^2 b_{s_K} \cup \{pq\}) \subset b_{s_{K_{p,q}}}.
\]

- **Theorem B** (K.-Tran.) Suppose $K'$ is an iterated cable of an adequate knot $K$. Then $K'$ satisfies the slope conjecture.
The following result provides a relation between the boundary slopes of a knot and those of the cables of the knot.

**Theorem A** (Motegi-Takata; K-Tran) For every knot $K \subset S^3$ and $(p, q)$ co-prime integers we have

$$(q^2bs_K \cup \{pq\}) \subset bs_{K_{p,q}}.$$

**Theorem B** (K.-Tran.) Suppose $K'$ is an iterated cable of an adequate knot $K$. Then $K'$ satisfies the slope conjecture.

An iterated torus knot is an iterated cable of the trivial knot. Hence iterated torus knots satisfy the slope conjecture.
The slope conjecture for cable knots

- The following result provides a relation between the boundary slopes of a knot and those of the cables of the knot.

**Theorem A** (Motegi-Takata; K-Tran) For every knot $K \subset S^3$ and $(p, q)$ co-prime integers we have

$$ (q^2 bs_K \cup \{pq\}) \subset bs_{K_{p,q}}. $$

- **Theorem B** (K.-Tran.) Suppose $K'$ is an iterated cable of an adequate knot $K$. Then $K'$ satisfies the slope conjecture.

- An iterated torus knot is an iterated cable of the trivial knot. Hence iterated torus knots satisfy the slope conjecture.

- Remark: Motegi-Takata show that the slope conjecture is preserved under connected sums. This, together with Theorem B, implies that graph knots (knots whose hyperbolic volume is 0) satisfies the slope conjecture.
Two new conjectures about the degree of the CJP

Recall that for every knot $K \subset S^3$, there is an integer $N_K > 0$ and periodic functions $a_K(n), b_K(n), c_K(n)$ such that

$$d_+[J_K(n)] = a_K(n)n^2 + b_K(n)n + c_K(n)$$

for $n \geq N_K$.

Conjecture 1 (K-Tran) For every non-trivial knot $K \subset S^3$, we have $b_K(n) \leq 0$.

Note that $b_U(n) = 1/2$ for the trivial knot $U$.

Conjecture 2 (Strong slope conjecture) [Kalfagianni-T.] Let $K$ be a knot and $r/s \in js_K$, with $s > 0$ and $\gcd(r, s) = 1$, a Jones slope of $K$. Then there is an essential surface $S \subset M_K$ with boundary slope $r/s$, and such that $\chi(S) | \partial S | s \in \{2b_K(n)\}'$. 

E. Kalfagianni
Two new conjectures about the degree of the CJP

- Recall that for every knot $K \subset S^3$, there is an integer $N_K > 0$ and periodic functions $a_K(n), b_K(n), c_K(n)$ such that

$$d_+[J_K(n)] = a_K(n) n^2 + b_K(n)n + c_K(n)$$

for $n \geq N_K$.

- **Conjecture 1 (K-Tran)** For every non-trivial knot $K \subset S^3$, we have

$$b_K(n) \leq 0.$$
Two new conjectures about the degree of the CJP

- Recall that for every knot $K \subset S^3$, there is an integer $N_K > 0$ and periodic functions $a_K(n), b_K(n), c_K(n)$ such that
  \[ d_+[J_K(n)] = a_K(n) n^2 + b_K(n)n + c_K(n) \]
  for $n \geq N_K$.
- **Conjecture 1** (K-Tran) For every non-trivial knot $K \subset S^3$, we have
  \[ b_K(n) \leq 0. \]
- Note that $b_U(n) = 1/2$ for the trivial knot $U$. 
Two new conjectures about the degree of the CJP

- Recall that for every knot $K \subset S^3$, there is an integer $N_K > 0$ and periodic functions $a_K(n), b_K(n), c_K(n)$ such that
  \[ d_+[J_K(n)] = a_K(n)n^2 + b_K(n)n + c_K(n) \]
  for $n \geq N_K$.
- **Conjecture 1 (K-Tran)** For every non-trivial knot $K \subset S^3$, we have
  \[ b_K(n) \leq 0. \]
- Note that $b_U(n) = 1/2$ for the trivial knot $U$.
- **Conjecture 2 [Strong slope conjecture] (Kalfagianni-T.)** Let $K$ be a knot and $r/s \in js_K$, with $s > 0$ and $\gcd(r, s) = 1$, a Jones slope of $K$. Then there is an essential surface $S \subset M_K$ with boundary slope $r/s$, and such that
  \[ \frac{\chi(S)}{|\partial S|} \in \{2b_K(n)\}'. \]
Some data on the degree of the CJP

| $K$ | $js_K$ | $\{2b_K(n)\}'$ | $\chi(S)$ | $|\partial S|$ |
|-----|--------|-----------------|-----------|----------|
| 8_{19} | $\{12\}$ | $\{0\}$ | 0 | 2 |
| 8_{20} | $\{8/3\}$ | $\{-1, -5/3\}$ | $-3$ | 1 |
| 8_{21} | $\{1\}$ | $\{-2\}$ | $-4$ | 2 |
| 9_{42} | $\{6\}$ | $\{-1\}$ | $-2$ | 2 |
| 9_{43} | $\{32/3\}$ | $\{-1, -5/3\}$ | $-3$ | 1 |
| 9_{44} | $\{14/3\}$ | $\{-2, -8/3\}$ | $-6$ | 1 |
| 9_{45} | $\{1\}$ | $\{-2\}$ | $-4$ | 2 |
| 9_{46} | $\{2\}$ | $\{-1\}$ | $-2$ | 2 |
| 9_{48} | $\{11\}$ | $\{-3\}$ | $-6$ | 2 |

Table: Non-alternating Montesinos knots up to nine crossings.

$\triangleright$ $s =$ denominator of Jones slope.

$$\frac{\chi(S)}{|\partial S|s} \in \{2b_K(n)\}'. $$
Consider the pretzel knot $K_p = (-2, 3, p)$, where $p \geq 5$ is an odd integer. Then

$$4a_{K_p}(n) = \frac{2(p^2 - p - 5)}{(p - 3)} \quad \text{and} \quad 2b_{K_p}(n) = -\frac{(p - 5)}{(p - 3)}.$$
Consider the pretzel knot \( K_p = (-2, 3, p) \), where \( p \geq 5 \) is an odd integer. Then

\[
4a_{K_p}(n) = \frac{2(p^2 - p - 5)}{p - 3} \quad \text{and} \quad 2b_{K_p}(n) = -\frac{p - 5}{p - 3}.
\]

\( K_p \) has an essential surface with boundary slope \( \frac{2(p^2 - p - 5)}{p - 3} \), with two boundary components, and Euler characteristic \( -(p - 5) \), which is equal to \( (p - 3)(2b_{K_p}(n)) \).
Consider the pretzel knot $K_p = (-2, 3, p)$, where $p \geq 5$ is an odd integer. Then

$$4a_{K_p}(n) = \frac{2(p^2 - p - 5)}{p - 3} \quad \text{and} \quad 2b_{K_p}(n) = -\frac{(p - 5)}{(p - 3)}.$$ 

$K_p$ has an essential surface with boundary slope $2(p^2 - p - 5)/(p - 3)$, with two boundary components, and Euler characteristic $-(p - 5)$, which is equal to $(p - 3)(2b_{K_p}(n))$.

Conjecture 1, is verified for all knots for which the Slopes conjecture is verified and all of their cables (K.- Tran).
Pretzel knots Example and concluding remarks:

Consider the pretzel knot $K_p = (-2, 3, p)$, where $p \geq 5$ is an odd integer. Then

$$4a_{K_p}(n) = \frac{2(p^2 - p - 5)}{p-3} \quad \text{and} \quad 2b_{K_p}(n) = -\frac{(p-5)}{p-3}.$$

$K_p$ has an essential surface with boundary slope $\frac{2(p^2 - p - 5)}{(p-3)}$, with two boundary components, and Euler characteristic $-(p-5)$, which is equal to $(p-3)(2b_{K_p}(n))$.

Conjecture 1, is verified for all knots for which the Slopes conjecture is verified and all of their cables (K.- Tran).

Conjecture 2 is verified for all semi adequate knots and their cables, graph knots, $(-2, 3, p)$-pretzel knots and their cables (K.- Tran).
Some References

Some References

Some References

Some References

Some References

Some References